Integer and Combinatorial Optimization: Mixed Integer Nonlinear Programming

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General form

Our general form is

\[
\begin{align*}
\min_{x,y} & \quad f(x, y) \\
\text{subject to} & \quad g_i(x, y) \leq 0 \quad \forall i \in I \\
x & \in X, \quad y \in Y
\end{align*}
\]

where \( X \) denotes a continuous set of variables and \( Y \) denotes a set of variables constrained to be integer.

The functions \( f(x, y) \) and \( g_i(x, y), i \in I, \) are general functions.

Typically, they are often assumed to be smooth (that is, their second derivative exists and is continuous).

In general, an MINLP is far harder to solve than a mixed integer linear program. Various approaches have been proposed.
Example: linear in $y$, convex in $x$

\[
\begin{align*}
\text{min} & \quad \frac{1}{2}(x - 9)^2 - 2y \\
\text{subject to} & \quad \frac{1}{2}x^2 + y \leq 5 \\
& \quad 0 \leq x \leq 5, \quad y \text{ binary}
\end{align*}
\]

If we relax the binary requirement, the optimal solution has $x = 3$, $y = \frac{1}{2}$. Thus, it is necessary to branch.
Example: linear in $y$, concave in $x$

$$\begin{align*}
\min & \quad -\frac{1}{2}x^2 + 2y \\
\text{subject to} & \quad -(x - 2)^2 - y \leq -1 \\
& \quad 0 \leq x \leq 2, \quad y \text{ binary}
\end{align*}$$

Two local minimizers to the NLP relaxation:
(i) $x = 1, y = 0$
(ii) $x = 2, y = 1$.

Thus, we may need to find all local minimizers of the NLP relaxation.

For this problem, if we relax the nonlinear constraint to $x - y \leq 1$, the point $x = 2, y = 1$ is no longer a local minimizer.
Example: linear in \( y \), concave in \( x \) (again)

\[
\begin{align*}
\text{min} & \quad -\frac{1}{2}(x - 1)^2 + 2y \\
\text{subject to} & \quad -(x - 4)^2 - y \leq -1 \\
& \quad 0 \leq x \leq 4, \quad y \text{ binary}
\end{align*}
\]

Three local minimizers to the NLP relaxation:

(i) \( x = 0, \ y = 0, \ \text{value} = -\frac{1}{2} \),  
(ii) \( x = 3, \ y = 0, \ \text{value} = -2 \),  
(iii) \( x = 4, \ y = 1, \ \text{value} = -2\frac{1}{2} \).

Thus, we may need to find all local minimizers of the NLP relaxation.

For this problem, even if we use the convex envelope and relaxing the nonlinear constraint to \( x - y \leq 3 \), the three local minimizers are still KKT points.
Spatial branching

We may partition the feasible region to get a global optimum. We can put envelopes on functions to get lower bounds.

This is a technique used in the package BARON by Sahinidis and Tawarmalani.

\[ g_i(x, y) \]

**domain of interest**

**envelope**
Spatial branching

We may partition the feasible region to get a global optimum. We can put envelopes on functions to get lower bounds.

This is a technique used in the package BARON by Sahinidis and Tawarmalani.