Logical Benders Decomposition for Quadratic Programs with Complementarity Constraints and Binary Variables[1]

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1. Introduction
   - QPCC
   - Simplified Method
2. Master Problem
   - How to select p?
3. Cut Strengthening
   - ℓ_1-norm sparsification
   - Tree guided sparsification
   - Hybrid Sparsification
4. Numerical Results
5. Extensions
   - Binary Variables
6. Summary & Future Research

Outline

Complementarity constraints and binary variables

We are interested in problems with both complementary variables and binary variables.

Eg: bilevel programs with binary upper level decision variables.
Complementarity arises from KKT optimality conditions for the lower level problem.

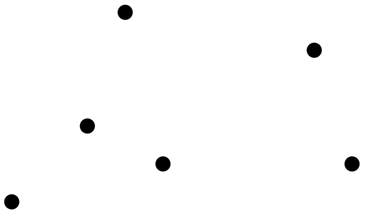
For example:
A facility location problem, with multiple competing shippers.

Previously: we've developed logical Benders approaches to linear and quadratic programs with complementarity constraints.
In this talk: Extend to also handle binary variables.

The emphasis in this talk will be on **improved methods for handling the complementarity constraints**.

Biofuel supply chain design (Bai, Ouyang, Pang)

Possible locations for fuel processing.
Fixed cost for opening:
binary variables.
Limited capacity.



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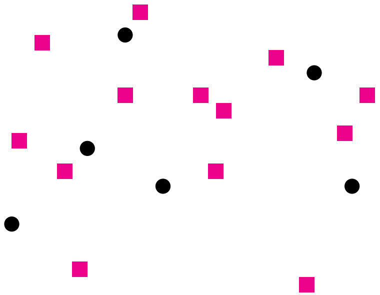

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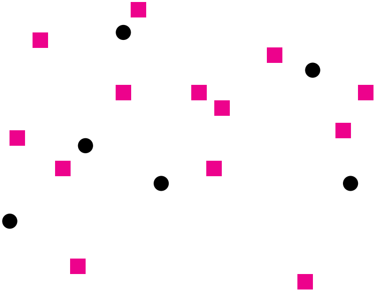

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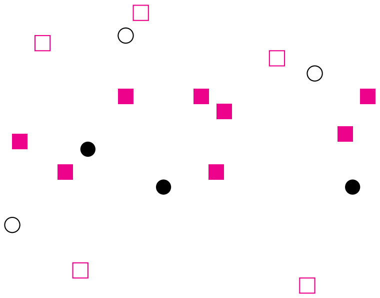

Bilevel programs

Bilevel program:

$$\begin{aligned} \min_{x,y} \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \le 0 \\ & y \text{ binary} \\ & x \in \operatorname{argmin}_x \{h(x) : \\ & \qquad r(x) \le v(y)\} \end{aligned}$$

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DC-MPCC:

$$\begin{aligned} \min_{x,y,z} \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \le 0 \\ & y \text{ binary} \\ & \nabla h(x) + \nabla r(x) z = 0 \\ & 0 \le v(y) - r(x) \perp z \ge 0 \end{aligned}$$

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Equivalent if $h(x)$ and $r(x)$ are convex, and a constraint qualification holds for the subproblems.

Our Standard Form Optimization Problem

$$\begin{array}{rrcl} \min_{x,y,w} & g^T x + x^T Q x & & \\ \text{s.t.} & A_I\, x + B_I\, y + C_I\, w & \geq & b_I \\ & A_E x + B_E y + C_E w & = & b_E \\ & 0 \leq y \perp w \geq 0 & & \end{array} \tag{QPCC}$$

- $y, w \in \mathbb{R}^{n_c}$ and $Q \in \mathbb{R}^{n_x \times n_x}$ is a positive semi-definite matrix.

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- We will discuss the addition of binary variables later.

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  - Explores complementarity pieces.
  - Discards pieces by generating cuts.

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& & & A_E x + B_E y + C_E w & = & b_E & (\ \mu_E\ ) \\
& & & w_i & \leq & 0,\ i : p_i = 0 & (\ \lambda_i^w\ ) \\
& & & y_i & \leq & 0,\ i : p_i = 1 & (\ \lambda_i^y\ ) \\
& & & w, y & \geq & 0. &
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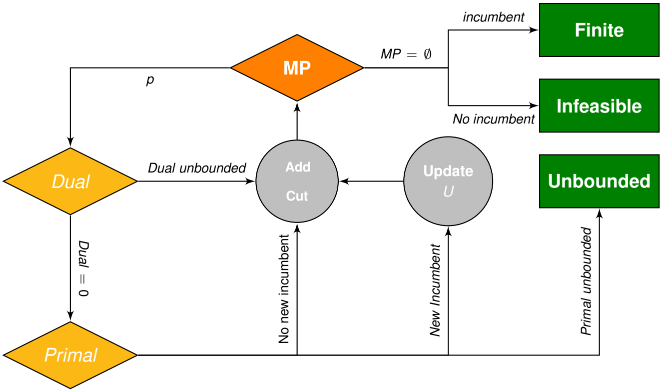
& & y_i & \le & 0,\ i : p_i = 1 & (\,\lambda_i^y\,) \\
& & w, y & \ge & 0. &
\end{array}$$

Therefore, (QPCC) is equivalent to solving $\min\limits_{p \in \{0,1\}^{n_c}} \phi_P(p)$.

Outline

Logical Benders Decomposition on QPCC - Simplified Algorithm



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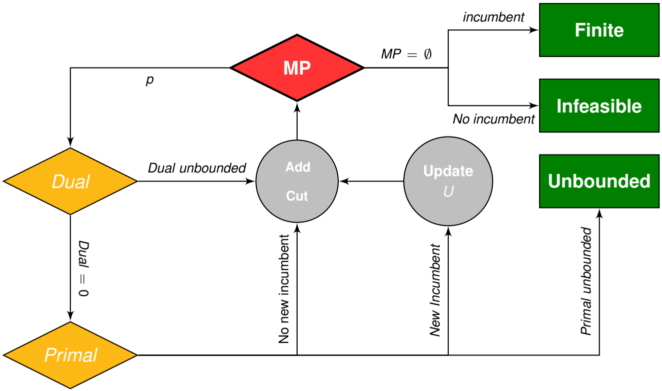

Complementarity pieces p are selected in a master problem.

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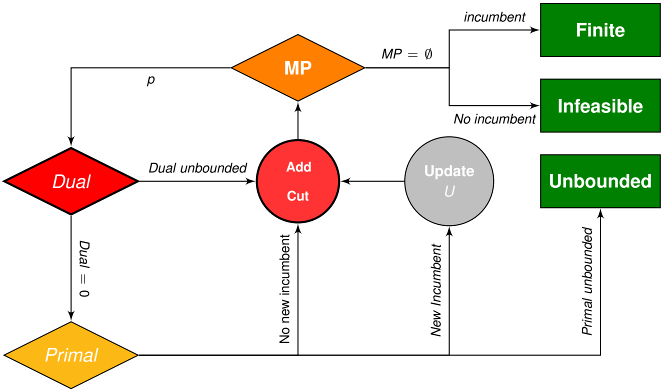

If the dual piece is unbounded, add cut to discard piece by infeasibility.
Cuts have the form

$$\sum_{i \in C^w} p_i + \sum_{i \in C^y} (1 - p_i) \geq 1.$$

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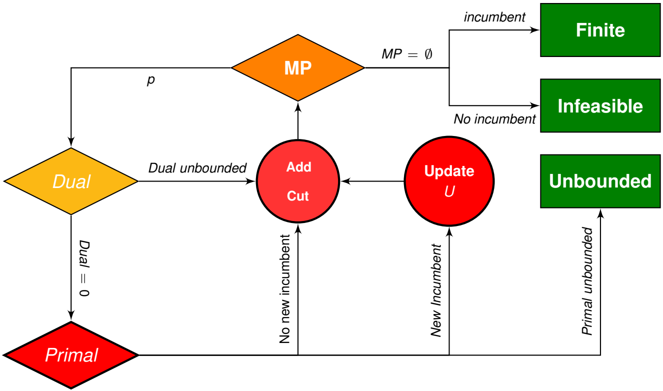
Logical Benders Decomposition on QPCC - Simplified Algorithm



If primal is finite, update incumbent and add cut to discard piece.
Cuts also have the form

$$\sum_{i \in C^w} p_i + \sum_{i \in C^y} (1 - p_i) \geq 1.$$

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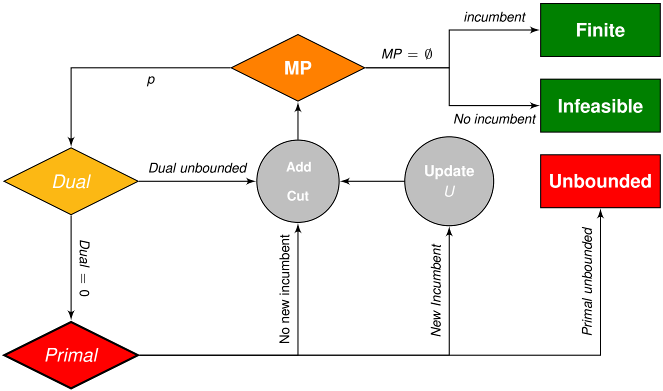

If primal is unbounded, QPCC is unbounded. STOP!

Outline

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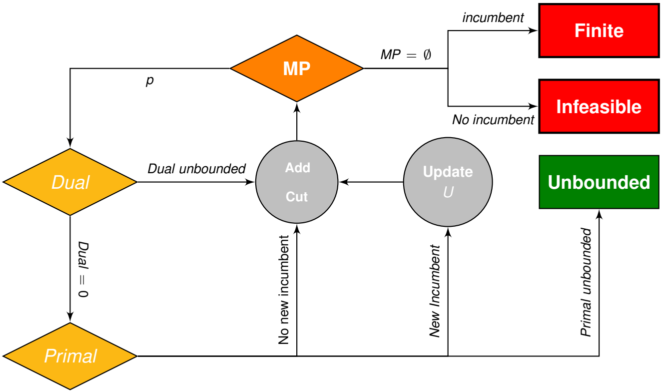

Method continues until master problem is infeasible.

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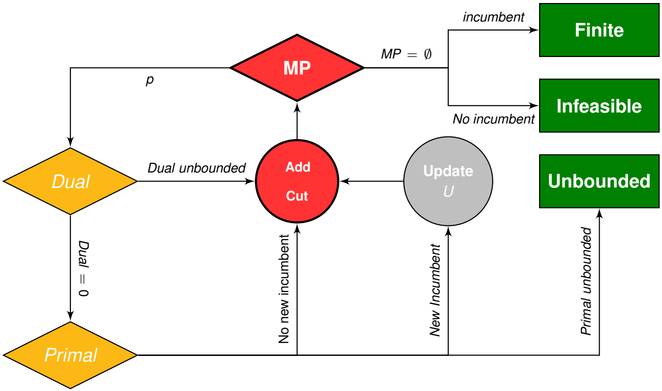

This talk:

- Piece selection (MP)
- Cut strengthening (Add Cut)

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- Suppose we are solving QPCC via a **Branch-and-Bound** approach.

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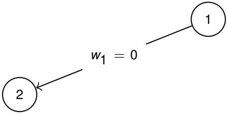

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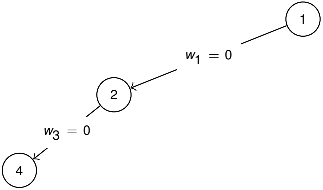

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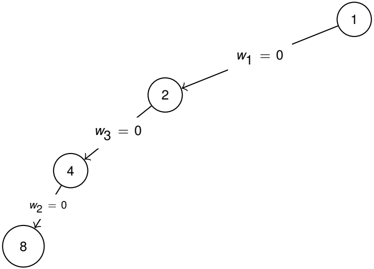

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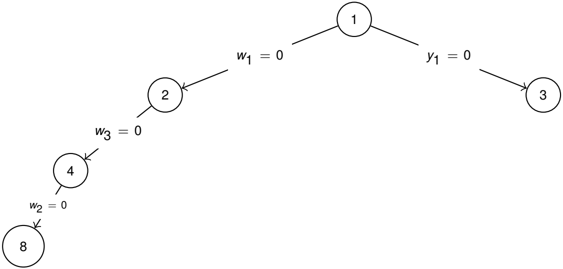

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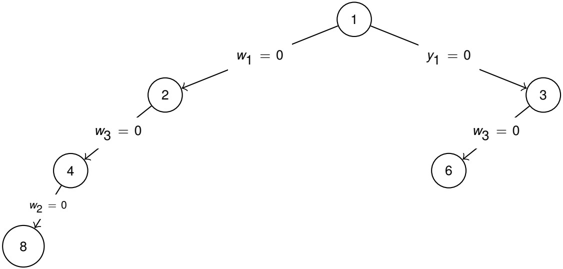

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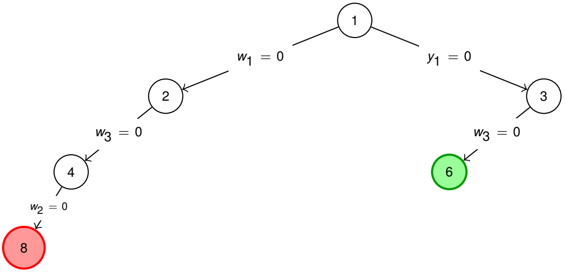
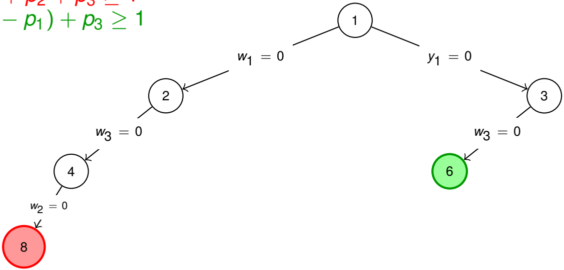
- Two nodes have been fathomed.



How to select p?

- Suppose we are solving QPCC via a **Branch-and-Bound** approach.
- Two nodes have been fathomed.
- The fathomed nodes may be interpreted as cuts.

  (Recall: $p_i = 0 \leftrightarrow w_i = 0$, $p_i = 1 \leftrightarrow y_i = 0$)
  - $p_1 + p_2 + p_3 \geq 1$
  - $(1 - p_1) + p_3 \geq 1$



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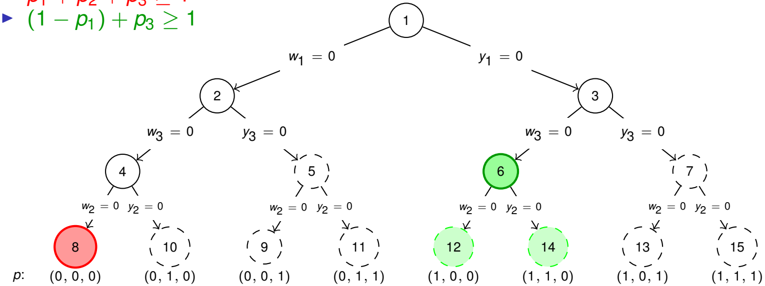

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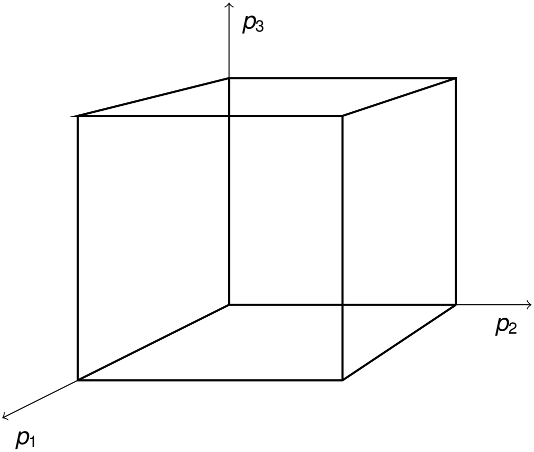

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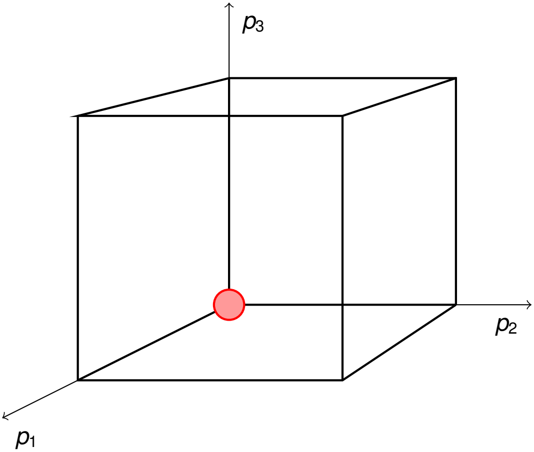

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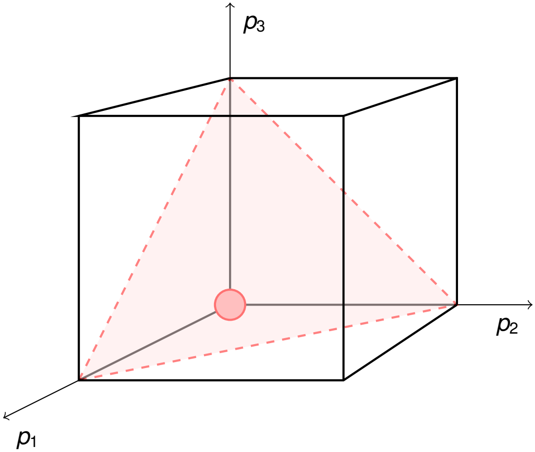

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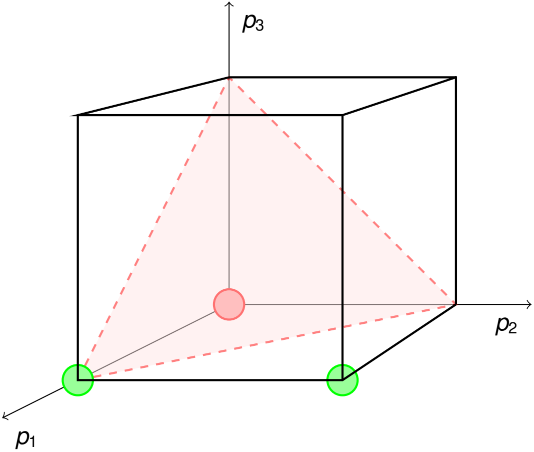

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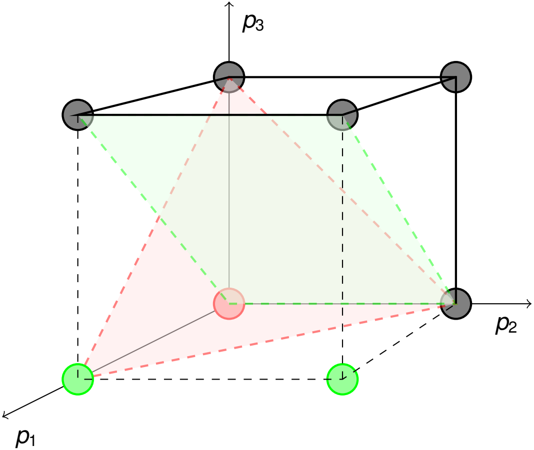

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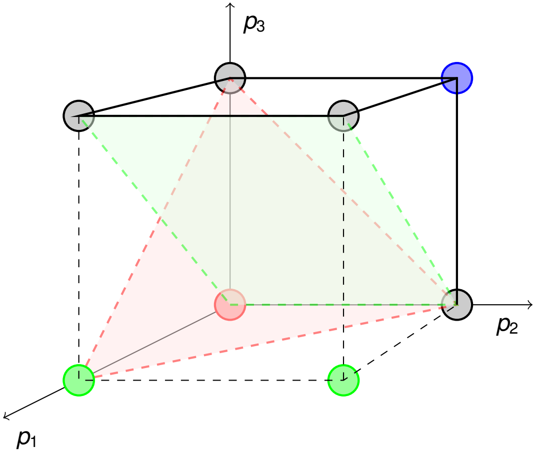

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Try to diversify search

Outline of Master Problem heuristic

Idea: “**Construct** a tree” from a **set of cuts** and pick a **leaf** p from a branch to increase likelihood of being **fathomed close to the root**. We are trying to **diversify** the search.

Outline of Master Problem heuristic

Input: A set of cuts $\{C_k^w, C_k^y\}$.
Output: A leaf $\hat{p}$, a path P.

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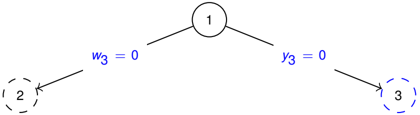
- Branch on most explored component $\hat{\jmath}$.
  $\hat{\jmath} = \arg\max_j \left|\{k : j \in C_k^w \cup C_k^y\}\right|$



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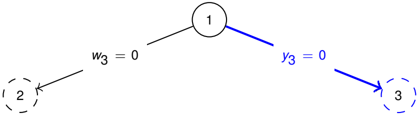
- Branch on most explored component $\hat{\jmath}$.
- Choose least explored side of branch.
  $\hat{p}_{\hat{\jmath}} = 0$ if $|\{k : \hat{\jmath} \in C_k^w\}| \leq \left|\{k : \hat{\jmath} \in C_k^y\}\right|$; $\hat{p}_{\hat{\jmath}} = 1$ otherwise.



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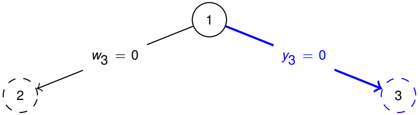
- Branch on most explored component $\hat{\jmath}$.
- Choose least explored side of branch.
- Add $\hat{\jmath}$ to P. Remove $\hat{\jmath}$ from branching candidates.



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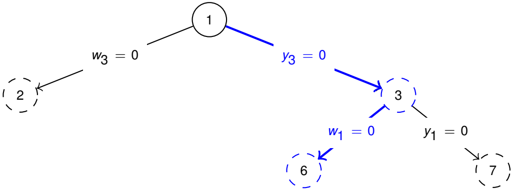

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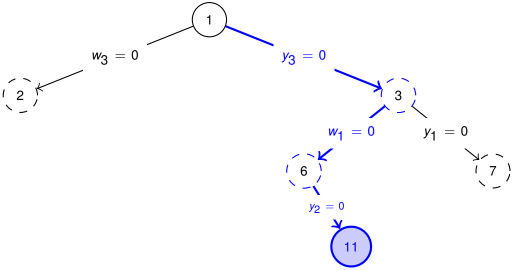

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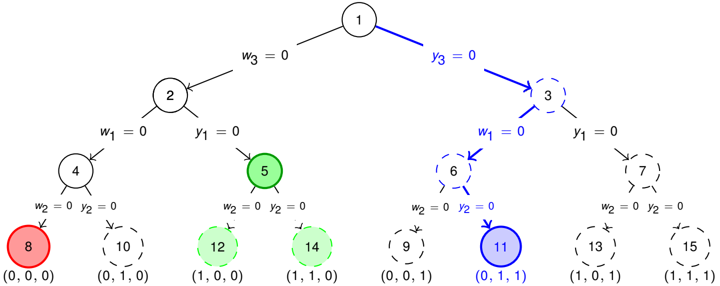

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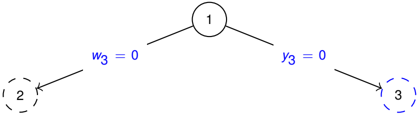

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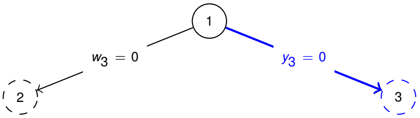

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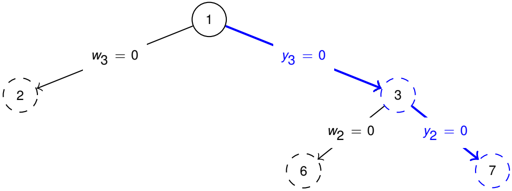

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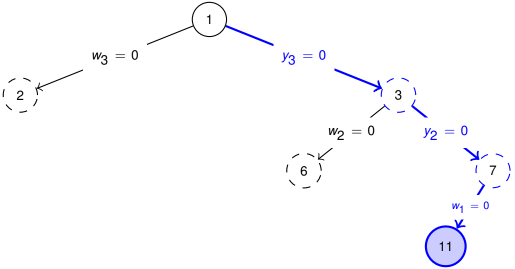

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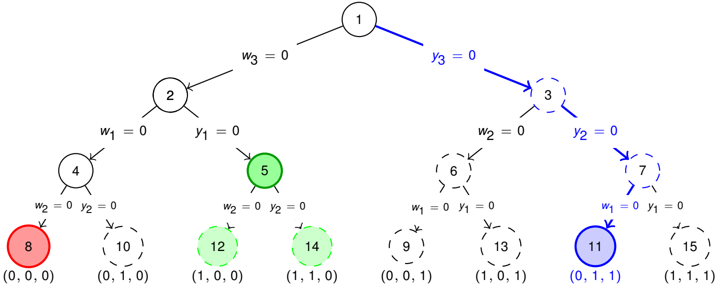

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- Would like to find smallest sets C^w and C^y such that $\phi_P(C,p) \geq U$.

Weighted ℓ_1-norm sparsification (WL1)

- Formulate an ℓ_0-norm dual problem

$$
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\phi_{D_s}(p) = & \min\limits_{\mu_I,\, \mu_E,\, \lambda^w,\, \lambda^y} & \left\|\lambda^w\right\|_0 + \left\|\lambda^y\right\|_0 & \\
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& & -B_I^T\mu_I + B_E^T\mu_E - \lambda^y & \leq 0 \\
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Remarks

- To solve ℓ_1 we use a weighted iterative procedure.

$$
\omega_i^{k+1} = \frac{1}{\max\{\lambda_{i,k+1}^w, \varepsilon\}} \text{ and } \gamma_i^{k+1} = \frac{1}{\max\{\lambda_{i,k+1}^y, \varepsilon\}}
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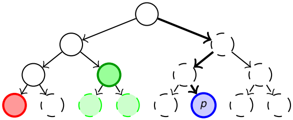
Tree guided sparsification (Tree)

We have piece *p* and path *P*.



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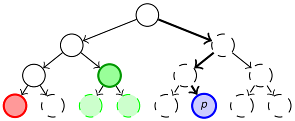

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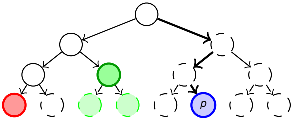

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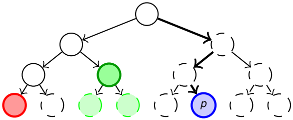

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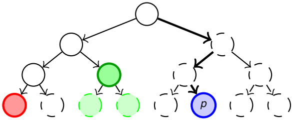

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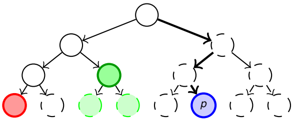

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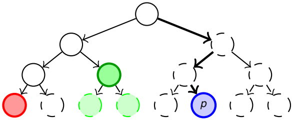

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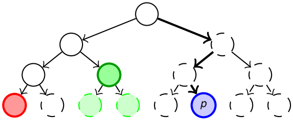

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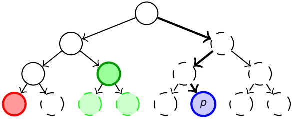

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Resulting cut is less likely to contain complementarities that are closer to the leaf p.

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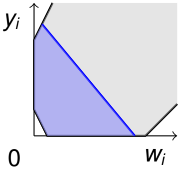

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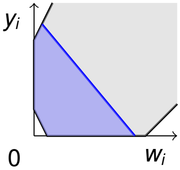

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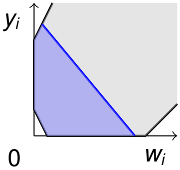

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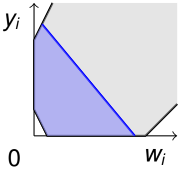

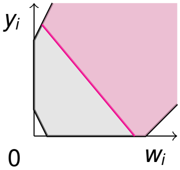

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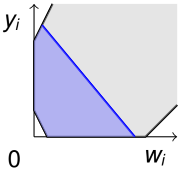

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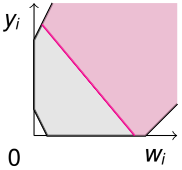

- A region bounded away from the origin.

  Add constraint $\mathbf{1}^T y + \mathbf{1}^T w \geq T$.

  Solve using logical Benders approach.

  Can function as a "certificate of optimality".

When use dual LPs to look for ray cuts, can add linearizations of the objective to the primal constraints.

Outline

Numerical Results

- LPCC:

n_c	Total wall clock time (s) LBD (2008)	WL1	Hyb	Tree
100	386.57	11.32	13.65	31.82
	41.98	2.28	2.53	4.97
	25.63	0.62	0.72	2.16
	961.65	9.39	6.61	24.8
	46.78	6.23	6.48	30.39
	32.18	1.58	1.74	3.81
	45.08	28.9	27.27	88.2
	488.18	4.38	3.74	10.51
	14.89	39.49	34.49	97.75
	811.55	8.55	13.58	40.56
300	190.92	132.69	241.58	637.43
	$>$ 1 hour	101.62	92.36	203.1
	528.83	1936.81	1805.64	$>$ 1 hour
	$>$ 1 hour	135.11	352.3	815.3
	553.83	343.34	295.64	919.32
	$>$ 1 hour	213.85	$>$ 1 hour	$>$ 1 hour
	0	264.25	235.96	686.39
	112.72	159.03	207.6	790.72
	872.64	569.24	387.07	992.7
	$>$ 1 hour	144.85	221.86	774.39

MIQP preprocessing, $T = 100$.

Numerical Results

- LPCC:

n_c	Total Iterations WL1	Hyb	Tree
100	34	34	33
	7	7	5
	2	2	2
	29	16	24
	20	18	31
	5	5	4
	88	75	92
	13	10	10
	120	91	97
	25	34	42
300	107	171	174
	82	63	51
	1391	1113	1047
	100	235	234
	271	216	251
	150	> 3000	> 3000
	200	170	197
	115	147	212
	378	255	271
	101	148	194

MIQP preprocessing, $T = 100$.

Numerical Results

- QPCC

n_c	Total Iterations			
	No outer box		outer Box	
	SAT	Tree	SAT	Tree
50	17	15	17	15
	38	22	2	5
	34	17	4	4
	30	19	8	3
	30	20	6	4
	37	17	25	12
	$>$100	37	$>$100	23
	$>$100	15	18	14
	38	23	38	23
	$>$100	8	8	5
100	137	35	130	34
	-	62	-	62
	102	23	102	15
	66	37	66	37
	7	6	7	5
	-	16	-	16
	-	8	-	6
	-	34	-	34
	47	3	47	3
	-	37	-	20

MIQP preprocessing, Hybrid, $T = 1000$.

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- Both ℓ_1 and tree-guided sparsification remain the same over this extension.

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