Logical Benders Decomposition

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Linear Programs with Complementarity Constraints (LPCCs)

Linear programs with complementarity constraints (LPCCs) arise, for example, in the modeling of bilevel problems.

In such a problem, the constraints include the requirement that some of the variables should be optimal solutions to another problem.

For example, when designing a traffic network, the designer needs to account for the fact that individual travelers will usually try to find an optimal solution to their own shortest path problems.

The lower level optimization problem can be modeled using its KKT optimality conditions under certain assumptions, which leads to complementarity constraints between slack variables and dual multipliers.

For more examples of LPCCs, see [3].
A framework for LPCCs

LPCCs can be solved using a variant of Benders decomposition. We consider the LPCC

\[
\begin{align*}
\min_{x, y, w} & \quad c^T x + d^T y + g^T w \\
\text{subject to} & \quad A x + B y + C w = b \\
& \quad 0 \leq y \perp w \geq 0 \\
& \quad x \geq 0
\end{align*}
\]

(1)

where \( x \in \mathbb{R}^n, y, w \in \mathbb{R}^m, b \in \mathbb{R}^k \), and all other vectors and matrices are dimensioned appropriately.

The orthogonality constraint between \( y \) and \( w \) gives the problem a combinatorial flavor:

for each component \( i \in \{1, \ldots, m\} \), either set \( y_i = 0 \) or set \( w_i = 0 \).
When bounds are available

To simplify the presentation, we assume the LP relaxation of (1) is feasible.
If we know upper bounds on each component of \( y \) and \( w \), we can construct an equivalent mixed integer program to (1):

\[
\begin{align*}
\min_{x, y, w, z} & \quad c^T x + d^T y + g^T w \\
\text{subject to} & \quad Ax + By + Cw = b \\
& \quad 0 \leq y \leq D_1 z \\
& \quad 0 \leq w \leq D_2 (e - z) \\
& \quad x \geq 0 \\
& \quad z \in \mathbb{B}^m
\end{align*}
\]

(2)

where \( D_1 \) and \( D_2 \) are diagonal matrices whose diagonal entries contain the upper bounds on \( y \) and \( w \) respectively, and \( e \) is a vector of ones.
If bounds are not known, we can use a logical Benders decomposition approach to solve the problem. We again use binary variables $z \in \mathbb{B}^m$, and we use them to enforce the complementarity restriction. The LPCC (1) can be represented equivalently as

$$
\begin{align*}
\min_{z \in \mathbb{B}^m} \varphi(z) &:= \min_{x, y, w} c^T x + d^T y + g^T w \\
\text{subject to} & \quad Ax + By + Cw = b \\
& \quad y_i \leq 0 \quad \text{if } z_i = 0 \\
& \quad w_i \leq 0 \quad \text{if } z_i = 1 \\
& \quad x, y, w \geq 0
\end{align*}
$$

(3)
Dual to subproblem

By LP duality, $\varphi(z)$ can be expressed equivalently as:

$$\min_{z \in \mathbb{B}^m} \varphi(z) = \max_{\lambda, \mu, \nu} \begin{cases} b^T \lambda \\ A^T \lambda \leq c \\ B^T \lambda - \mu \leq d \\ C^T \lambda - \nu \leq g \\ \mu_i = 0 \text{ if } z_i = 1 \\ \nu_i = 0 \text{ if } z_i = 0 \\ \mu, \nu \geq 0 \end{cases}$$

since the primal constraint $y_i \leq 0$ only exists if $z_i = 0$, so the corresponding dual variable $\mu_i$ can only be nonzero if $z_i = 0$, or equivalently $\mu_i = 0$ if $z_i = 1$. 


Algorithmic framework

In the algorithmic framework, we have a Master Problem to pick $z$. We then solve the subproblem (4) to find $\varphi(z)$ and find constraints to add to the Master Problem and restrict the choice of $z$.

Let $(\hat{\lambda}, \hat{\mu}, \hat{\nu})$ be an optimal dual solution to the subproblem for a particular $\hat{z}$, so $\varphi(\hat{z}) = b^T \hat{\lambda}$.

If this is the best $z$ seen so far, then we store $\hat{z}$ as the incumbent solution. Initially, there may be no incumbent solution, so the first feasible $z$ becomes the first incumbent.
Constraint to add to Master Problem

We want to minimize $\varphi(z)$, so we want to restrict attention to choices of $z$ that are better than $\hat{z}$.

Any $z$ that allows the dual feasible solution $(\hat{\lambda}, \hat{\mu}, \hat{\nu})$ will have value at least $\varphi(\hat{z})$, so we add a constraint on $z$ to rule this out:

$$\sum_{i: \hat{\mu}_i > 0} z_i + \sum_{i: \hat{\nu}_i > 0} (1 - z_i) \geq 1. \quad (5)$$

This constraint forces either $z_i = 0$ for some $i$ with $\hat{\nu}_i > 0$ or $z_i = 1$ for some $i$ with $\hat{\mu}_i > 0$.

For such a $z$, the constraints in the subproblem would either force $\nu_i = 0$ for some $i$ with $\hat{\nu}_i > 0$ or force $\mu_i = 0$ for some $i$ with $\hat{\mu}_i > 0$. 
Ray cuts

Similar cuts can be defined when (4) has an \textbf{unbounded optimal value}, using a ray.

The positive components of the ray lead to exactly the same constraint as (5) for the Master Problem.

In practice, it is important to try to \textit{sparsify} the constraint, in order to make it more powerful.

This can be done by requiring additional components of $\mu$ and/or $\nu$ be zero in (4), or by branching on the $z$ variables.

The sparsest constraint (5) would involve only one variable, which implies that we can fix that variable.
The Master Problem is a Satisfiability problem, since the constraints (5) are satisfiability constraints.

For more details on logical Benders decomposition for LPCCs, see [2]. For extension to quadratic programs with complementarity constraints, see [1].
A numerical example

We use the following example to illustrate the algorithm.

minimize \((x, y, w)\)

subject to

\[
\begin{align*}
2x_1 + x_2 + 2y_1 - y_3 \\
x_1 + x_2 - x_3 &= 5 \\
-x_1 + y_3 + w_1 &= 1 \\
-x_2 - y_1 - y_2 + w_2 &= 0 \\
-x_1 - x_2 + y_2 + w_3 &= 2 \\
x &\geq 0 \\
0 \leq y &\perp w \geq 0
\end{align*}
\]

(6)
Subproblem dual

If we choose $z = (1, 1, 1)$ we get the dual LP of the form (4) as follows:

\[
\begin{align*}
\text{max}_{\lambda, \mu, \nu} & \quad 5\lambda_1 + \lambda_2 + 2\lambda_4 \\
\text{subject to} & \quad \lambda_1 - \lambda_2 - \lambda_4 \leq 2 \\
& \quad \lambda_1 - \lambda_3 - \lambda_4 \leq 1 \\
& \quad -\lambda_1 \leq 0 \\
& \quad -\lambda_3 - \mu_1 \leq 2 \\
& \quad -\lambda_3 + \lambda_4 - \mu_2 \leq 0 \\
& \quad \lambda_2 - \mu_3 \leq -1 \\
& \quad -\lambda_2 - \nu_1 \leq 0 \\
& \quad -\lambda_3 - \nu_2 \leq 0 \\
& \quad \lambda_4 - \nu_3 \leq 0 \\
& \quad \mu_1 = 0 \\
& \quad \mu_2 = 0 \\
& \quad \mu_3 = 0 \\
& \quad \mu, \nu \geq 0
\end{align*}
\]
Solution to subproblem dual

This problem is unbounded and has a ray:

\[ d_\lambda = (0, 0, 1, 1), \quad d_\mu = (0, 0, 0), \quad d_\nu = (0, 1, 1). \]  \( (8) \)

Hence from (5), we obtain the valid constraint:

\[ (1 - z_2) + (1 - z_3) \geq 1. \]  \( (9) \)
Subproblem dual

One choice of \( z \) that satisfies this constraint is \( z = (1, 0, 1) \). The dual problem is then

\[
\begin{align*}
\max_{\lambda, \mu, \nu} & \quad 5\lambda_1 + \lambda_2 + 2\lambda_4 \\
\text{subject to} & \quad \lambda_1 - \lambda_2 - \lambda_4 \leq 2 \\
& \quad \lambda_1 - \lambda_3 - \lambda_4 \leq 1 \\
& \quad -\lambda_1 \leq 0 \\
& \quad -\lambda_3 - \lambda_4 \leq 2 \\
& \quad -\lambda_3 + \lambda_4 \leq 0 \\
& \quad \lambda_2 - \mu_1 \leq 2 \\
& \quad -\lambda_3 + \lambda_4 - \mu_2 \leq 0 \\
& \quad \lambda_2 - \mu_3 \leq -1 \\
& \quad \lambda_3 - \nu_1 \leq 0 \\
& \quad -\lambda_3 + \lambda_4 - \nu_2 \leq 0 \\
& \quad \lambda_4 - \nu_3 \leq 0 \\
& \quad \mu_1 = 0 \\
& \quad \nu_2 = 0 \\
& \quad \mu_3 = 0 \\
& \quad \mu, \nu \geq 0
\end{align*}
\]
Solution to subproblem dual

This problem is unbounded and has a ray:

\[ d_\lambda = (0, 0, 0, 1), \quad d_\mu = (0, 1, 0), \quad d_\nu = (0, 0, 1). \quad (11) \]

Hence from (5), we obtain the valid constraint:

\[ z_2 + (1 - z_3) \geq 1. \quad (12) \]
Subproblem dual

One choice of $z$ satisfying both (9) and (12) is $z = (0, 0, 0)$. This gives the dual LP

$$\max_{\lambda, \mu, \nu} \quad 5\lambda_1 + \lambda_2 + 2\lambda_4$$

subject to

$$\begin{align*}
\lambda_1 & - \lambda_2 & - \lambda_4 & \leq 2 \\
\lambda_1 & - \lambda_3 & - \lambda_4 & \leq 1 \\
-\lambda_1 & \leq 0 \\
-\lambda_3 & - \mu_1 & \leq 2 \\
-\lambda_3 & + \lambda_4 & - \mu_2 & \leq 0 \\
\lambda_2 & - \mu_3 & \leq -1 \\
\lambda_2 & - \nu_1 & \leq 0 \\
\lambda_3 & - \nu_2 & \leq 0 \\
\lambda_4 & - \nu_3 & \leq 0 \\
\nu_1 & = 0 \\
\nu_2 & = 0 \\
\nu_3 & = 0 \\
\mu, \nu & \geq 0
\end{align*}$$

(13)
Solution to subproblem dual

This problem has an optimal solution with value 5:

$$\lambda = (1, 0, 0, 0), \quad \mu = (0, 0, 1), \quad \nu = (0, 0, 0)$$  \hspace{1cm} (14)

so from (5), we obtain the valid constraint:

$$z_3 \geq 1.$$ \hspace{1cm} (15)

The choice \( z = (0, 0, 0) \) becomes the **incumbent** best solution found.
Updated Master Problem

The satisfiability Master Problem has three constraints:

\( (1 - z_2) + (1 - z_3) \geq 1 \) \hspace{1cm} (16)
\( z_2 + (1 - z_3) \geq 1 \) \hspace{1cm} (17)
\( z_3 \geq 1 \) \hspace{1cm} (18)

These constraints are inconsistent, so we can stop, and conclude that the best solution found is optimal.
Solution to LPCC

Thus, the optimal solution is $z = (0, 0, 0)$. With this choice of $z$, the primal problem (3) has the form

$$\begin{align*}
\text{minimize} \quad & 2x_1 + x_2 + 2y_1 - y_3 \\
\text{subject to} \quad & x_1 + x_2 - x_3 = 5 \\
& -x_1 + y_3 + w_1 = 1 \\
& -x_2 - y_1 - y_2 + w_2 = 0 \\
& -x_1 - x_2 + y_2 + w_3 = 2 \\
& x, y, w \geq 0 \\
& y_1 = y_2 = y_3 = 0
\end{align*}$$

(19)

The optimal solution is

$$x = (0, 5, 0), \quad y = (0, 0, 0), \quad w = (1, 5, 7),$$

(20)

achieving the value 5.
Logical Benders


Logical Benders

On convex quadratic programs with linear complementarity constraints.

On the global solution of linear programs with linear complementarity constraints.

J. Hu, J. E. Mitchell, J.S. Pang, and B. Yu.
On linear programs with linear complementarity constraints.
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