Two Examples of Lagrangian Relaxation of Integer Programs

This is based on material in Chapter 4 of [1]. We look at cutting plane methods to solve the Lagrangian dual, which are column generation methods in the dual of the dual.

1 Vehicle Routing Problems with Time Windows

Customers: $C := \{1, \ldots, n\}$. For each $i \in C$, have demand $d_i$, and a time window $[a_i, b_i]$ for the delivery.

Vehicles: $V := \{1, \ldots, v\}$ identical vehicles with capacity $q$. All vehicles start and end their routes at the same depot; $N$ is the set of all nodes, which consists of the customer nodes $C$ plus two copies of the depot (labelled nodes 0 and $n + 1$). Depot has trivially satisfied values for its time window. Cost to traverse arc $(i, j)$ is $c_{ij}$, with $c_{0,n+1} = 0$. Can arrive at customer $i$ before time $a_i$ but must then wait until time $a_i$. Parameter $t_{ij}$ is the time required to traverse the link from customer $i$ to customer $j$. Parameter $T_{ij} = \max\{t_{ij} + b_i - a_j, 0\}$.

Variables:

$x_{ijk} = \begin{cases} 1 & \text{if vehicle } k \text{ visits customer } i \text{ immediately before customer } j \\ 0 & \text{otherwise} \end{cases}$

$s_{ik} = \begin{cases} \text{departure time} & \text{if vehicle } k \text{ visits customer } i \\ \text{anything otherwise} & \end{cases}$

Formulation:

\[
\begin{align*}
\text{min} & \quad \sum_{k \in V} \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ijk} \\
\text{subject to} & \quad \sum_{k \in V} \sum_{j \in N} x_{ijk} = 1 \quad \forall i \in C \\
& \quad \sum_{i \in C} d_i \sum_{j \in N} x_{ijk} \leq q \quad \forall k \in V \\
& \quad \sum_{j \in N} x_{0jk} = 1 \quad \forall k \in V \\
& \quad \sum_{i \in N} x_{ihk} - \sum_{j \in N} x_{hjk} = 0 \quad \forall h \in C, \forall k \in V \\
& \quad \sum_{i \in N} x_{i,n+1,k} = 1 \quad \forall k \in V \\
& \quad s_{ik} + t_{ij} - T_{ij}(1 - x_{ijk}) \leq s_{jk} \quad \forall i, j \in N, \forall k \in V \\
& \quad a_i \leq s_{ik} \leq b_i \quad \forall i \in N, \forall k \in V \\
& \quad x_{ijk} \text{ binary} \quad \forall i, j \in N, \forall k \in V.
\end{align*}
\]
Lagrangian relaxation:

Constraint (1b) is the only one where the left hand side involves a sum over the vehicles. So if we dualize this constraint, we will be left with a separable problem. Let $\lambda$ denote the vector of Lagrangian multipliers, with one component for each $i \in C$. To make the notation easier, we extend $\lambda$ to include two additional components corresponding to nodes 0 and $n + 1$ and set those components equal to zero. Since the vehicles are identical, each subproblem can be regarded as a problem for a single vehicle. The part of the Lagrangian relaxation corresponding to a single vehicle can be written:

$$\min \sum_{i \in N} \sum_{j \in N} (c_{ij} - \lambda_i) x_{ij}$$
subject to
\begin{align*}
\sum_{i \in C} d_i \sum_{j \in N} x_{ij} &\leq q \\
\sum_{j \in N} x_{0j} &= 1 \\
\sum_{i \in N} x_{ih} - \sum_{j \in N} x_{hj} &= 0 &\forall h \in C \\
\sum_{i \in N} x_{i,n+1} &= 1 \\
s_i + t_{ij} - T_{ij}(1 - x_{ij}) &\leq s_j &\forall i, j \in N \\
a_i &\leq s_i \leq b_i &\forall i \in N \\
x_{ij} \text{ binary} &\forall i, j \in N.
\end{align*}

This is known as an elementary shortest path problem with resource constraints.

If we use a cutting plane approach to solve the Lagrangian dual, then the dual to the LP cutting plane approximation has variables corresponding to vehicle subtours that satisfy the time windows and only visit a subset of the customers. The constraints are that every customer should appear in exactly one subtour.

Lagrangian dual (disaggregated):

$$\max_{\theta, \lambda} \quad v \theta + \sum_{i \in C} \lambda_i$$
subject to
\begin{align*}
\theta + \sum_{i \in C} \sum_{j \in N} \lambda_i x_{ij}^p &\leq b^p &\forall \text{subtours } p \in P \\
\sum_{i \in N} x_{ij}^p &\leq 1 &\forall i \in N
\end{align*}

Subtours $P$ are generated as needed. $b^p$ is an appropriate right hand side. Dual of the dual:

$$\min_{\pi} \quad \sum_{p \in P} b^p \pi^p$$
subject to
\begin{align*}
\sum_p \pi^p &= v \\
\sum_{p \in P} \sum_{j \in N} x_{ij}^p \pi^p &= 1 &\forall i \in C \\
\pi^p &\geq 0 &\forall p \in P
\end{align*}

For more details, see [2, 1].
2 Capacitated lot sizing with setup times

Time periods $T := \{1, \ldots, n\}$, items $M := \{1, \ldots, m\}$, all processed on a single machine with capacity $C_t$ at time $t$. Have demand $d_{jt}$ for item $j$ at time $t$. Production, holding, setup costs of item $j$ at time $t$ are $c_{jt}$, $h_{jt}$, and $f_{jt}$, respectively. In any time period, the machine can perform many tasks; if it produces item $j$ in time period $t$ it must be setup in that time period for that item. Processing and setup times for item $j$ in time $t$ are $a_{jt}$ and $b_{jt}$ respectively. Let $D_{jt} = \sum_{i=t}^{n} d_{ji}$, the remaining demand for item $j$.

Variables:

- $x_{jt}$: production of item $j$ in time $t$
- $s_{jt}$: inventory of item $j$ at end of time period $t$
- $y_{jt}$: binary variable equal to one if and only if item $j$ is produced in time period $t$

Formulation:

\[
\begin{align*}
\min_{x,y,s} \quad & \sum_{t \in T} \sum_{j \in M} (c_{jt} x_{jt} + h_{jt} s_{jt} + f_{jt} y_{jt}) \\
\text{subject to} \quad & \sum_{j \in M} (a_{jt} x_{jt} + b_{jt} y_{jt}) \leq C_t \quad \forall t \in T \\
& s_{jt} - s_{jt-1} + x_{jt} = d_{jt} + s_{jt} \quad \forall j \in M, \forall t \in T \\
& x_{jt} \leq D_{jt} y_{jt} \quad \forall j \in M, \forall t \in T \\
& x_{jt} \geq 0 \quad \forall j \in M, \forall t \in T \\
& s_{jt} \geq 0 \quad \forall j \in M, \forall t \in T \\
& y_{jt} \text{ binary} \quad \forall j \in M, \forall t \in T 
\end{align*}
\]

Lagrangian relaxation:

The items are linked through constraint \((3b)\), so we dualize this constraint, with multipliers $\lambda_t$ for each $t \in T$. The resulting Lagrangian relaxation is separable by item, and the subproblem for item $j$ is the following single item lot-sizing problem:

\[
\begin{align*}
\min_{x,y,s} \quad & \sum_{t \in T} \left((c_{jt} + a_{jt} \lambda_t) x_{jt} + h_{jt} s_{jt} + (f_{jt} + b_{jt} \lambda_t) y_{jt}\right) \\
\text{subject to} \quad & s_{jt-1} + x_{jt} = d_{jt} + s_{jt} \quad \forall t \in T \\
& x_{jt} \leq D_{jt} y_{jt} \quad \forall t \in T \\
& x_{jt} \geq 0 \quad \forall t \in T \\
& s_{jt} \geq 0 \quad \forall t \in T \\
& y_{jt} \text{ binary} \quad \forall t \in T 
\end{align*}
\]

Let $z_{LR_j}(\lambda)$ denote the optimal value of this problem, for each item $j \in M$.  

Lagrangian dual

The Lagrangian dual problem can be written

\[ \max_{\lambda \geq 0} \left\{ -\sum_{t \in T} C_t \lambda_t + \sum_{j \in M} z_{LR_j}(\lambda) \right\}. \]

In a cutting plane approach, we introduce variables \( \theta_j \) for each \( z_{LR_j}(\lambda) \). We have iterates \( \lambda^p \), for \( p = 1, 2, \ldots \), with solutions \((x^p_{jt}, y^p_{jt}, s^p_{jt})\) to the Lagrangian relaxation for each \( j \in M \). We relax the dual problem as

\[ \max_{\lambda, \theta} \quad -\sum_{t \in T} C_t \lambda_t + \sum_{j \in M} \theta_j \]

subject to

\[ \begin{align*}
\theta_j &\leq z_{LR_j}(\lambda^p) + \sum_{t \in T} (\lambda_t - \lambda^p_t) \left( a_{jt} x^p_{jt} + b_{jt} y^p_{jt} \right), & j \in M, \ p = 1, 2, \ldots \\
\lambda &\geq 0
\end{align*} \]

or equivalently

\[ \max_{\lambda, \theta} \quad -\sum_{t \in T} C_t \lambda_t + \sum_{j \in M} \theta_j \]

subject to

\[ \begin{align*}
\theta_j - \sum_{t \in T} \left( \lambda_t (a_{jt} x^p_{jt} + b_{jt} y^p_{jt}) \right) &\leq b^p_j := z_{LR_j}(\lambda^p) - \sum_{t \in T} \left( \lambda^p_t (a_{jt} x^p_{jt} + b_{jt} y^p_{jt}) \right), & j \in M, \ p = 1, 2, \ldots \\
\lambda &\geq 0
\end{align*} \]

Each \( p \) corresponds to a production plan that is feasible for each of the single lot sizing problems. The dual of the dual is

\[ \min_{\pi} \quad \sum_{j \in M} \sum_{p=1, 2, \ldots} b^p_j \pi^p_j \]

subject to

\[ \begin{align*}
\sum_{j \in M} \sum_{p=1, 2, \ldots} (a_{jt} x^p_{jt} + b_{jt} y^p_{jt}) \pi^p_j &\leq C_t & t \in T \\
\sum_{p=1, 2, \ldots} \pi^p_j & = 1 & j \in M \\
\pi^p_j &\geq 0 & j \in M, \ p = 1, 2, \ldots
\end{align*} \]

Here, \( \pi^p_j \) is the proportion of the total production of item \( j \) that is met using production plan \( p \). \( b^p_j \) is the cost of meeting all production needs for item \( j \) using plan \( p \).

For more details, see [3, 1].

References

