An example of solving a Lagrangian Dual Problem

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2014
Consider the convex nonlinear optimization problem

\[
\begin{align*}
\min & \quad x_1^2 + 2x_2^2 + 2x_1 + 8x_2 \\
\text{s.t.} & \quad g(x_1, x_2) := -x_1 - 2x_2 + 10 \leq 0 \\
& \quad x \geq 0
\end{align*}
\]

We will take \( X = \{x \geq 0\} \), and place the other constraint in the Lagrangian:

\[
L(x_1, x_2, u) := x_1^2 + 2x_2^2 + 2x_1 + 8x_2 + u(10 - x_1 - 2x_2).
\]
Thus, the dual function is given by

$$\Theta(u) := \min_{x \geq 0} x_1^2 + 2x_2^2 + 2x_1 + 8x_2 + u(10 - x_1 - 2x_2)$$

and the dual problem is

$$\max_{u \geq 0} \Theta(u).$$

We are going to exploit the fact that $\Theta(u)$ is a concave function by creating a **piecewise-linear** approximation to it.
Construct initial piecewise linear approximation

Initially, try $u = 0$:

$$\Theta(0) = \min_{x \geq 0} x_1^2 + 2x_2^2 + 2x_1 + 8x_2 = 0,$$

achieved at $x_1 = x_2 = 0$. The subgradient gives an overestimate of $\Theta(u)$. The slope is given by the function value $g(0, 0) = 10$, so we get the inequality:

$$\Theta(u) \leq \Theta(0) + 10(u - 0) = 0 + 10u = 10u. \quad (1)$$
Construct initial piecewise linear approximation

To try to get an upper bound on the optimal value of $u$, we will try $u = 20$. We get

$$\Theta(20) = \min_{x \geq 0} x_1^2 + 2x_2^2 + 2x_1 + 8x_2 + 20(10 - x_1 - 2x_2)$$

$$= \min_{x \geq 0} x_1^2 + 2x_2^2 - 18x_1 - 32x_2 + 200$$

$$= \min_{x \geq 0} (x_1 - 9)^2 + 2(x_2 - 8)^2 - 81 - 128 + 200$$

$$= -9, \quad \Theta(u) \leq \Theta(20) + g(\bar{x})^T(u - 20)$$

achieved at $x_1 = 9$, $x_2 = 8$. The subgradient gives an overestimate of $\Theta(u)$. The slope is given by the function value $g(9, 8) = -15$, so we get the inequality:

$$\Theta(u) \leq \Theta(20) - 15(u - 20) = 291 - 15u. \quad (2)$$
Solve current relaxation of dual problem

We now use equations (1) and (2) to choose a new $u$, by solving the linear programming problem:

$$\begin{align*}
\text{max} & \quad z \\
\text{s.t.} & \quad z \leq 10u \\
& \quad z \leq 291 - 15u.
\end{align*}$$
The optimal solution is where the two lines intersect, that is, where $10u = 291 - 15u$, so $u = 291/25 = 11\frac{16}{25}$. 
Find new valid subgradient inequality

In order to make the arithmetic a little easier, we’ll take \( u = 12 \) as our next estimate:

\[
\Theta(12) = \min_{x \geq 0} x_1^2 + 2x_2^2 + 2x_1 + 8x_2 + 12(10 - x_1 - 2x_2)
\]

\[
= \min_{x \geq 0} x_1^2 + 2x_2^2 - 10x_1 - 16x_2 + 120
\]

\[
= \min_{x \geq 0} (x_1 - 5)^2 + 2(x_2 - 4)^2 - 25 - 32 + 120
\]

\[
= 63,
\]

achieved at \( x_1 = 5, x_2 = 4 \).
New subgradient inequality

The subgradient gives an overestimate of $\Theta(u)$. The slope is given by the function value $g(5, 4) = -3$, so we get the inequality:

$$\Theta(u) \leq \Theta(12) - 3(u - 12) = 99 - 3u.$$  \hspace{1cm} (3)

Note also that $\Theta(12) = 63$ gives an underestimate for the optimal value of the dual problem.
Updated relaxation of Lagrangian dual

We now use equations (1)–(3) to choose a new $u$, by solving the linear programming problem:

\[
\begin{align*}
\text{max } & \quad z \\
\text{s.t. } & \quad z \leq 10u \\
& \quad z \leq 291 - 15u \\
& \quad z \leq 99 - 3u.
\end{align*}
\]
Solution to relaxation

The optimal solution is where the first and third constraints intersect, that is, where $10u = 99 - 3u$, so $u = 99/13 = \frac{10}{13}$. 
Find new valid subgradient inequality

In order to make the arithmetic a little easier, we’ll take $u = 8$ as our next estimate:

$$ \Theta(8) = \min_{x \geq 0} x_1^2 + 2x_2^2 + 2x_1 + 8x_2 + 8(10 - x_1 - 2x_2) $$

$$ = \min_{x \geq 0} x_1^2 + 2x_2^2 - 6x_1 - 8x_2 + 80 $$

$$ = \min_{x \geq 0} (x_1 - 3)^2 + 2(x_2 - 2)^2 - 9 - 8 + 80 $$

$$ = 63, $$

achieved at $x_1 = 3, x_2 = 2$. 
New subgradient inequality

The subgradient gives an overestimate of $\Theta(u)$. The slope is given by the function value $g(3, 2) = 3$, so we get the inequality:

$$\Theta(u) \leq \Theta(8) + 3(u - 8) = 39 + 3u.$$  (4)

Note also that $\Theta(8) = 63$ gives an underestimate for the optimal value of the dual problem.
We now use equations (1)–(4) to choose a new $u$, by solving the linear programming problem:

\[
\begin{align*}
\text{max} & \quad z \\
\text{s.t.} & \quad z \leq 10u \\
& \quad z \leq 291 - 15u \\
& \quad z \leq 99 - 3u \\
& \quad z \leq 39 + 3u.
\end{align*}
\]
The optimal solution is at the intersection of the last two constraints, that is, when $99 - 3u = 39 + 3u$, or $u = 10$. 
Find new valid subgradient inequality

We have

\[ \Theta(10) = \min_{x \geq 0} x_1^2 + 2x_2^2 + 2x_1 + 8x_2 + 10(10 - x_1 - 2x_2) \]

\[ = \min_{x \geq 0} x_1^2 + 2x_2^2 - 8x_1 - 12x_2 + 100 \]

\[ = \min_{x \geq 0} (x_1 - 4)^2 + 2(x_2 - 3)^2 - 16 - 18 + 100 \]

\[ = 66, \]

achieved at \( x_1 = 4, x_2 = 3 \). The function value \( g(4, 3) = 0 \) gives a subgradient of \( \Theta(10) \), so \( u = 10 \) must be optimal. Since the problem is convex, we obtain that \( x_1 = 4, x_2 = 3 \) is the optimal primal solution. Checking, we obtain that the primal value is indeed 66 at this point and it is feasible.
Final approximation to Lagrangian dual function

\[ z = 66 + 0u \]