1 Assignment problem with budget constraint

We consider the assignment problem with a budget constraint

\[
\max_{x \in \mathbb{R}^{n^2}} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}
\]

subject to

\[
\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j
\]

\[
\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i
\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij} \leq b
\]

\(x\) binary

Several different Lagrangian relaxations are possible.

1.1 Relax the budget constraint

If we relax the final constraint, the Lagrangian relaxation is the assignment problem

\[
\max_{x \in \mathbb{R}^{n^2}} \quad b \lambda + \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - \lambda t_{ij}) x_{ij}
\]

subject to

\[
\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j
\]

\[
\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i
\]

\(x\) binary

Since the Lagrangian relaxation is an assignment problem, it can be solved by solving its LP relaxation. Thus, for this approach, the optimal value of the Lagrangian dual is only as good as the value of the LP relaxation of (1).

1.2 Relax the assignment constraints

If we relax the assignment constraints, we introduce two sets of Lagrangian multipliers: \(\lambda\) for the first set of constraints and \(\mu\) for the second set. The Lagrangian relaxation is a knapsack problem:

\[
\max_{x \in \mathbb{R}^{n^2}} \quad \sum_{j=1}^{n} \lambda_{j} + \sum_{i=1}^{n} \mu_{i} + \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - \lambda_{j} - \mu_{i}) x_{ij}
\]

subject to

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij} \leq b
\]

\(x\) binary
Since the Lagrangian relaxation cannot be solved just by solving its LP relaxation, we should obtain a better bound, so the optimal value of the Lagrangian dual should be larger than the optimal value of the LP relaxation of (1). The drawback is that it is harder to solve the LP relaxation.

Other relaxations are possible. The trade off is in the ease of solving the relaxation versus the strength of the bound.

2 Solving the Lagrangian dual problem

We let $z_{IP}$ denote the optimal value of the integer program

$$z_{IP} := \max_{x} c^T x$$
subject to $A^1 x \leq b^1$ (complicating constraints)
A^2 x \leq b^2$ (nice constraints)
$x \in \mathbb{Z}^n_+$

The matrices $A^1 \in \mathbb{R}^{m_1 \times n}$ and $A^2 \in \mathbb{R}^{m_2 \times n}$, and all vectors are dimensioned appropriately. For any $\lambda \in \mathbb{R}^{m_1}$, we obtain the Lagrangian relaxation

$$z_{LR}(\lambda) := \max_{x} c^T x + \lambda^T (b^1 - A^1 x)$$
subject to $A^2 x \leq b^2$
x $\in \mathbb{Z}^n_+$

The optimal value of problem (3) gives an upper bound on the optimal value of (2) for any $\lambda \geq 0$. In the Lagrangian dual problem, this upper bound is minimized:

$$z_{LD} := \min_{\lambda} z_{LR}(\lambda)$$
subject to $\lambda \geq 0$. (4)

Theorem 1 The function $z_{LR}(\lambda)$ is convex.

Proof. Let $Q$ denote the feasible region for (3). Note that this is the same for all $\lambda$. For simplicity, we assume $Q$ is bounded, so it contains a finite number of points $\{x^1, \ldots, x^q\}$. For any $\lambda$, we have

$$z_{LR}(\lambda) = \max_{x \in Q} \{c^T x + \lambda^T (b^1 - A^1 x)\} = \lambda^T b^1 + \max_{i=1,\ldots,q} \{ (c^T - \lambda^T A^1) x^i \}$$

which is the maximum of a finite number of linear functions, which is itself a piecewise linear convex function.

Theorem 2 Let $\hat{x}$ solve (3) for a given value $\hat{\lambda}$. Then

$$z_{LR}(\lambda) \geq z_{LR}(\hat{\lambda}) + (\lambda - \hat{\lambda})^T (b^1 - A^1 \hat{x})$$

so $b^1 - A^1 \hat{x}$ is a subgradient of the convex function $z_{LR}(\lambda)$ at $\lambda = \hat{\lambda}$.
Proof. This follows directly from the equality in the previous theorem. In particular, we have
\[ z_{LR}(\lambda) \geq c^T \hat{x} + \lambda^T (b^1 - A^1 \hat{x}) \quad \text{since } \hat{x} \in Q \]
\[ = c^T \hat{x} + \hat{\lambda}^T (b^1 - A^1 \hat{x}) + (\lambda - \hat{\lambda})^T (b^1 - A^1 \hat{x}) \]
\[ = z_{LR}(\hat{\lambda}) + (\lambda - \hat{\lambda})^T (b^1 - A^1 \hat{x}) \]
as required. □

Since we have a subgradient for the convex function \( z_{LR}(\lambda) \), we can minimize it using various different convex optimization algorithms. For example, we can use subgradient descent methods.

Alternatively, we can use cutting plane methods, using the inequality from Theorem 2. Each time we solve the Lagrangian relaxation, we get a solution \( x^p \). We thus can approximate the Lagrangian dual function using the linear program:

\[
\begin{align*}
\min_{\theta, \lambda} & \quad \theta \\
\text{subject to} & \quad \theta \geq c^T x^p + \lambda^T (b^1 - A^1 x^p) \quad \text{for } p = 1, 2, \ldots \\
& \quad \lambda \geq 0
\end{align*}
\]

For example, returning to the problem from the previous lecture, perhaps initially we have two points \( x^1 = (0, 3) \) and \( x^2 = (3, 2) \), giving us the following piecewise linear approximation:

This approximation is minimized by \( \hat{\lambda} = \frac{3}{7} \). The value of the approximation is \( \hat{\theta} = \frac{21}{7} \).

We then get
\[
\begin{align*}
z_{LR}(\frac{3}{7}) = \max_{x \in \mathbb{R}^2} & \quad x_1 + \frac{3}{7} (2 - 2x_1 + x_2) \\
\text{subject to} & \quad x_1 + 2x_2 \leq 8 \\
& \quad x_1 \leq 3 \\
& \quad x_2 \leq 3 \\
& \quad x \geq 0, \text{ integer}
\end{align*}
\]
The solution is $x = (2, 3)$ with value $z_{LR}(\frac{3}{7}) = 2\frac{3}{7} > \hat{\theta}$. The constraint added to the Lagrangian dual approximation is

$$
\theta \geq 2\frac{3}{7} + (\lambda - \frac{3}{7})(2 - 4 + 3) = 2\frac{3}{7} + (\lambda - \frac{3}{7}) = 2 + \lambda.
$$

The updated value of $\hat{\lambda} = \frac{1}{3}$ with the value of the approximation equal to $\hat{\theta} = 2\frac{1}{3}$. It can be verified that $z_{LR}(\hat{\lambda}) = \hat{\theta}$, so the algorithm can terminate at this point with the optimal solution to the Lagrangian dual.

When the Lagrangian relaxation is separable, it may be advantageous computationally to break up the scalar variable $\theta$ into a sum of variables $\theta_i$, each corresponding to one of the separated subproblems.