Examples of Duals of Semidefinite Programs

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We write our standard form semidefinite program as

\[
\begin{align*}
\min_X & \quad C \cdot X \\
\text{subject to} & \quad A_i \cdot X = b_i, \ i = 1, \ldots, m \\
& \quad X \succeq 0
\end{align*}
\]  

where:
- \(C\) is an \(n \times n\) symmetric matrix
- \(A_i\) is an \(n \times n\) symmetric matrix for \(i = 1, \ldots, m\)
- \(b_i\) is a scalar for \(i = 1, \ldots, m\).

The parameter matrices \(C\) and \(A_i\) need not be positive semidefinite, although they are assumed to be symmetric. Recall that \(C \cdot X\) represents the Frobenius inner product between the symmetric matrices \(C\) and \(X\), which is equal to the \(\text{trace}(CX)\). The dual problem is

\[
\begin{align*}
\max_{y \in \mathbb{R}^m, S \in S^n_+} & \quad b^T y \\
\text{subject to} & \quad \sum_{i=1}^m y_i A_i + S = C \\
& \quad S \succeq 0
\end{align*}
\]

Notice that the dual slack variables \(S = C - \sum_{i=1}^m y_i A_i\) constitute a symmetric positive semidefinite matrix in feasible dual solutions.

**Example 1.** We look at the SDP relaxation of the MaxCut problem on the following graph:

![Graph diagram]

The SDP relaxation is written

\[
\begin{align*}
\max_X & \quad 0.25 \text{trace}(L_G X) \\
\text{subject to} & \quad X_{ii} = 1 \quad \text{for} \ i = 1, \ldots, n \\
& \quad X \succeq 0,
\end{align*}
\]

where \(L_G\) is the Laplacian matrix. In our case, we have

\[
L_G = \begin{bmatrix}
9 & -6 & -3 \\
-6 & 7 & -1 \\
-3 & -1 & 4
\end{bmatrix}
\]
Turning the problem into an equivalent minimization problem and rescaling, the SDP relaxation of the MaxCut instance is equivalent to the problem

\[
\min_{X \in \mathbb{S}^3_+} \begin{bmatrix} -9 & 6 & 3 \\ 6 & -7 & 1 \\ 3 & 1 & -4 \end{bmatrix} \cdot X
\]

subject to

\[
\begin{align*}
X_{11} &= 1 \\
X_{22} &= 1 \\
X_{33} &= 1 \\
X &\preceq 0
\end{align*}
\]

so the dual is

\[
\max_{y \in \mathbb{R}^3, S \in \mathbb{S}^3_+} y_1 + y_2 + y_3
\]

subject to

\[
\begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} + S = \begin{bmatrix} -9 & 6 & 3 \\ 6 & -7 & 1 \\ 3 & 1 & -4 \end{bmatrix}
\]

\[
S \succeq 0
\]

The optimal solution to the MaxCut instance is to set \( V_1 = \{1\}, V_2 = \{2, 3\} \), corresponding to \( \bar{x} = (1, -1, -1)^T \) and then

\[
\bar{X} = \bar{x} \bar{x}^T = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}
\]

with value \(-36\). The optimal dual solution is \( y = (-18, -12, -6) \), with value \(-36\). It’s feasible, since the resulting dual slack matrix is

\[
S = \begin{bmatrix} -9 & 6 & 3 \\ 6 & -7 & 1 \\ 3 & 1 & -4 \end{bmatrix} - \begin{bmatrix} -18 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 3 \\ 6 & 5 & 1 \\ 3 & 1 & 2 \end{bmatrix}
\]

which has determinant 0, and all principal subdeterminants nonnegative.

Note that the matrix product \( SX \) is the zero matrix. This is the SDP version of complementary slackness.

**Theorem 1** (Strong duality). Assume the primal and dual semidefinite programs are both feasible. If there exists a feasible solution \((\bar{y}, \bar{S})\) to the dual SDP where \( \bar{S} \) is **positive definite** then the primal and dual optimal values agree and the primal optimal value is attained. If there exists a feasible solution \((\bar{X})\) to the primal SDP where \( \bar{X} \) is **positive definite** then the primal and dual optimal values agree and the dual optimal value is attained.

If there exists a feasible solution \((\bar{X})\) to the primal SDP where \( \bar{X} \) is positive definite and a feasible solution \((\bar{y}, \bar{S})\) to the dual SDP where \( \bar{S} \) is positive definite then the primal and dual optimal values agree and both optimal value are attained.

In SDP relaxations of combinatorial optimization problems, it is typical that that trace(\(X\)) must equal a constant. This ensures there exists a strictly dual feasible solution (provided the dual is feasible).
Example 2. An example with a duality gap:

\[
\begin{align*}
\min_{X \in S^3_+} & \quad X_{33} \\
\text{subject to} & \quad X_{22} = 0 \\
& \quad X_{12} + X_{21} + X_{33} = 1 \\
& \quad X \succeq 0
\end{align*}
\]

Note that the first equality constraint then forces \(X_{12} = X_{21} = 0\) in order for \(X \succeq 0\) to hold. Then every feasible solution must have \(X_{33} = 1\), so the optimal primal value is 1.

The dual problem is

\[
\max_{y \in \mathbb{R}^2, S \in S^3_+} y_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

so

\[
S = \begin{bmatrix} 0 & -y_2 & 0 \\ -y_2 & -y_1 & 0 \\ 0 & 0 & 1 - y_2 \end{bmatrix}
\]

The problem is feasible, for example we can take \(y = (0, 0)\). Since \(S_{11} = 0\), we must have \(y_2 = 0\), so the optimal dual value is 0.

Example 3. An example which is primal infeasible and has a finite dual optimal value:

\[
\begin{align*}
\min_{X \in S^3_+} & \quad 0 \\
\text{subject to} & \quad X_{11} = 0 \\
& \quad X_{12} + X_{21} = 2 \\
& \quad X \succeq 0
\end{align*}
\]

Since \(X_{11} = 0\), the requirement \(X \succeq 0\) forces \(X_{12} = X_{21} = 0\), but then the second linear constraint is violated.

The dual problem is

\[
\max_{y \in \mathbb{R}^2, S \in S^3_+} 2y_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

so

\[
S = \begin{bmatrix} -y_1 & -y_2 \\ -y_2 & 0 \end{bmatrix}
\]

Any feasible solution must have \(y_2 = 0\) so the dual has a finite optimal value of 0.
Example 4. Example 3 is close to a feasible primal problem:

$$\min_{X \in S^+_2} \quad 0$$
subject to
$$X_{11} = \epsilon$$
$$X_{12} + X_{21} = 2$$
$$X \succeq 0$$

where $\epsilon$ is a positive parameter. This has feasible solutions including

$$X = \begin{bmatrix} \epsilon & 1 \\ 1 & 1/\epsilon \end{bmatrix}$$

The dual problem is

$$\max_{y \in \mathbb{R}^2, S \in S^+_2} \quad \epsilon y_1 + 2y_2$$
subject to
$$y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$S \succeq 0$$

so

$$S = \begin{bmatrix} -y_1 & -y_2 \\ -y_2 & 0 \end{bmatrix}$$

Any feasible solution must have $y_2 = 0$ and $y_1 \leq 0$ so the dual has a finite optimal value of 0.

Example 5. An example with an irrational optimal value:

$$\min_{X \in S^+_2} \quad X_{12} + X_{21}$$
subject to
$$X_{11} = 1$$
$$X_{22} = 2$$
$$X \succeq 0$$

The optimal value is $-2\sqrt{2}$ and the optimal solution is

$$X = \begin{bmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}.$$

The dual problem is

$$\max_{y \in \mathbb{R}^2, S \in S^+_2} \quad y_1 + 2y_2$$
subject to
$$y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$S \succeq 0$$

so

$$S = \begin{bmatrix} -y_1 & 1 \\ 1 & -y_2 \end{bmatrix}$$

The optimal dual solution is $y = (-\sqrt{2}, -1/\sqrt{2})$ with optimal value of $-2\sqrt{2}$. 