Duals of Semidefinite Programs

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April 2018
Outline

1. SDP formulation
2. A Lagrangian relaxation of an SDP
3. The dual of an SDP
Semidefinite programs

The variables of a semidefinite program are arranged in a symmetric positive semidefinite $n \times n$ matrix.

There are linear constraints on the elements of the matrix, and an objective function that is also a linear function of the elements of the matrix.

We denote the matrix of variables by $X$.

The requirement that a matrix $X$ be symmetric and positive semidefinite is written $X \succeq 0$.

The set of symmetric positive semidefinite $n \times n$ matrix is denoted $\mathbb{S}_+^n$.

Note that $\mathbb{S}_+^n$ is a convex cone: any nonnegative combination of symmetric positive semidefinite matrices is also a symmetric positive semidefinite matrix.
Frobenius inner product
For example, with \( n = 3 \), we could have a linear constraint

\[
3X_{12} + 3X_{21} + 5X_{33} = 15. \tag{1}
\]

This constraint can be expressed using a Frobenius inner product between two matrices: given two symmetric \( n \times n \) matrices \( M \) and \( X \), their Frobenius inner product is

\[
\text{trace} (MX) = M \bullet X := \langle M, X \rangle := \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij}X_{ij}. \tag{2}
\]

Constraint (1) can then be written as

\[
\begin{bmatrix}
0 & 3 & 0 \\
3 & 0 & 0 \\
0 & 0 & 5
\end{bmatrix} \bullet X = 15. \tag{3}
\]
\[
\text{MaxCut:} \quad X_{ii} = 1, \quad A_i = \begin{bmatrix} 0 \cdots 0 \\ 0 & \cdots & 0 \end{bmatrix} \quad 1 \text{ in position } (i,i). \quad 0 \text{ elsewhere.}
\]

Constraint can be written:

\[
\text{Trace} (A_i X) = 1
\]

(\underline{Frobenius product:})

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} (A_i)_{jk} X_{jk}
\]
Node packing:

\[ \text{trace} (X) = 1 \]

\[ \text{trace} (I X) = 1. \]

\[ X_{ij} = 0 \text{ if edge } (i, j) \in E \]

Matrix \( A^{ij} \):

\[ (A^{ij})_{kl} = \begin{cases} 1 & \text{if } k = i, l = j \\ 1 & \text{if } k = j, l = i \\ 0 & \text{otherwise} \end{cases} \]

\[ 0 = \text{trace} (A^{ij} X) = X_{ij} + X_{ji} \]
Frobenius inner product and the trace

Lemma

For symmetric matrices $M$ and $X$, their Frobenius inner product satisfies $M \cdot X = \text{trace}(MX)$. 
Proof of lemma

The lemma states that the Frobenius inner product of two symmetric matrices is equal to the sum of the diagonal terms of their (usual) matrix product. We have

\[
\text{trace}(MX) = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} M_{ij}X_{ji} \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij}X_{ij} \quad \text{since } X \text{ is symmetric}
\]

\[
= M \bullet X
\]

from definition (2).
A property of the trace

This equivalence with the trace is useful for performing manipulations. Of particular importance is the following lemma, showing the order of the matrices can be changed.

**Lemma**

Let $M$ be a $p \times q$ matrix and let $R$ be a $q \times p$ matrix. Then $\text{trace}(MR) = \text{trace}(RM)$. 
Proof of lemma

The proof follows immediately from the definition:

\[
\text{trace}(MR) = \sum_{i=1}^{p} (MR)_{ii} = \sum_{i=1}^{p} \left( \sum_{j=1}^{q} M_{ij} R_{ji} \right)
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} R_{ji}
\]

\[
= \sum_{j=1}^{q} \left( \sum_{i=1}^{p} R_{ji} M_{ij} \right)
\]

\[
= \sum_{j=1}^{q} (RM)_{jj}
\]

\[
= \text{trace}(RM)
\]

as required.
Standard form SDP

We write our standard form semidefinite program as

\[
\begin{align*}
\min_X & \quad C \cdot X \\
\text{subject to} & \quad A_i \cdot X = b_i, \ i = 1, \ldots, m \\
& \quad X \succeq 0
\end{align*}
\]  

(4)

where:

- \( C \) is an \( n \times n \) symmetric matrix
- \( A_i \) is an \( n \times n \) symmetric matrix for \( i = 1, \ldots, m \)
- \( b_i \) is a scalar for \( i = 1, \ldots, m \).

The parameter matrices \( C \) and \( A_i \) need not be positive semidefinite, although they are assumed to be symmetric.
MaxCut:
- Edge weights $W$.
- Laplacian matrix $L = D - W$.
- Objective function: $\max \frac{1}{4} L \cdot X$

Node packing:
$\max \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \cdot X$
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A Lagrangian relaxation of an SDP

Lagrangian relaxation

We obtain a Lagrangian relaxation of (4) by relaxing the equality constraints, with multipliers $y_i, i = 1, \ldots, m$:

$$\theta(y) := \min_X C \cdot X + \sum_{i=1}^{m} y_i (b_i - A_i \cdot X)$$
subject to $X \succeq 0$  \hspace{1cm} (5)

Equivalently,

$$\theta(y) = b^T y + \min_X (C - \sum_{i=1}^{m} y_i A_i) \cdot X$$
subject to $X \succeq 0$  \hspace{1cm} (6)
Solving the Lagrangian relaxation

It is easy to find $\theta(y)$. In particular, we have

$$\theta(y) = \begin{cases} b^T y & \text{if } C - \sum_{i=1}^{m} y_i A_i \text{ is positive semidefinite} \\ -\infty & \text{if } C - \sum_{i=1}^{m} y_i A_i \text{ has at least one negative eigenvalue} \end{cases}$$

(7)

This is a consequence of the following pair of lemmas.
Negative eigenvalue

Lemma

Let $M$ be a symmetric $n \times n$ matrix. If $M$ is not positive semidefinite then there exists $X \in S_n^+$ with $M \cdot X < 0$.

Proof.

Let $v \in \mathbb{R}^n$ be an eigenvector of $M$ with negative eigenvalue. Define $X = vv^T \in S_n^+$. Then

\[
M \cdot X = \text{trace}(MX) = \text{trace}(Mvv^T) = \text{trace}(v^TMv) \quad \text{since} \quad \text{trace}(AB) = \text{trace}(BA) = v^TMv \quad \text{since} \quad v^TMv \text{ is a } 1 \times 1 \text{ matrix} < 0 \quad \text{from the definition of } v
\]

as required.
Negative eigenvalue

Lemma

Let $M$ be a symmetric $n \times n$ matrix. If $M$ is not positive semidefinite then there exists $X \in S^n_+$ with $M \cdot X < 0$.

Proof.

Let $v \in \mathbb{R}^n$ be an eigenvector of $M$ with negative eigenvalue. Define $X = vv^T \in S^n_+$. Then

$$M \cdot X = \text{trace}(MX)$$
$$= \text{trace}(Mvv^T)$$
$$= \text{trace}(v^T Mv) \quad \text{since trace}(AB) = \text{trace}(BA)$$
$$= v^T Mv \quad \text{since } v^T Mv \text{ is a } 1 \times 1 \text{ matrix}$$
$$< 0 \quad \text{from the definition of } v$$

as required.
Nonnegative inner product if psd

Lemma

Let $M$ be a symmetric $n \times n$ matrix. If $M$ is positive semidefinite then $M \cdot X \geq 0$ for all $X \in \mathbb{S}_+^n$. 
Proof of lemma

Since $M \in \mathbb{S}_+^n$, it has an eigendecomposition $M = UDU^T$, where each entry of the diagonal matrix $D$ is nonnegative. Denote the columns of $U$ by $u_i$, $i = 1, \ldots, n$, so the $u_i$ are the eigenvectors of $M$. Let $X \in \mathbb{S}_+^n$. We then have

$$M \cdot X = \text{trace}(MX) = \text{trace}(UDU^T X)$$

$$= \text{trace} \left( \left( \sum_{i=1}^{n} D_{ii} u_i u_i^T \right) X \right) = \text{trace} \left( \sum_{i=1}^{n} D_{ii} \left( u_i u_i^T X \right) \right)$$

$$= \sum_{i=1}^{n} D_{ii} \text{trace} \left( u_i u_i^T X \right) = \sum_{i=1}^{n} D_{ii} \text{trace} \left( u_i^T X u_i \right)$$

$$= \sum_{i=1}^{n} D_{ii} u_i^T X u_i \quad \text{since each } u_i^T X u_i \text{ is a } 1 \times 1 \text{ matrix}$$

$$\geq 0 \quad \text{since each } D_{ii} \geq 0 \text{ and } X \text{ is positive semidefinite}$$

as required.
Value of Lagrangian relaxation

The statement (7) then follows since:

- If $C - \sum_{i=1}^{m} y_i A_i$ is positive semidefinite then
  $(C - \sum_{i=1}^{m} y_i A_i) \cdot X \geq 0$ for all $X \in S^+_n$ from Lemma 4.
  Hence the optimal choice is $X = 0$, giving $\theta(y) = b^T y$.

- If $C - \sum_{i=1}^{m} y_i A_i$ has at least one negative eigenvalue with corresponding eigenvector $v$:
  We can take $X = tvv^T$ for any $t \geq 0$.
  As $t \to \infty$, we get $(C - \sum_{i=1}^{m} y_i A_i) \cdot X \to -\infty$. 
Value of Lagrangian relaxation

The statement (7) then follows since:

- If $C - \sum_{i=1}^{m} y_i A_i$ is positive semidefinite then 
  $(C - \sum_{i=1}^{m} y_i A_i) \cdot X \geq 0$ for all $X \in S^n_+$ from Lemma 4. 
  Hence the optimal choice is $X = 0$, giving $\theta(y) = b^T y$.

- If $C - \sum_{i=1}^{m} y_i A_i$ has at least one negative eigenvalue with 
  corresponding eigenvector $v$: 
  We can take $X = tvv^T$ for any $t \geq 0$. 
  As $t \to \infty$, we get $(C - \sum_{i=1}^{m} y_i A_i) \cdot X \to -\infty$. 
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The dual of an SDP

The dual problem

Since $\theta(y)$ is the value of the Lagrangian relaxation, it provides a lower bound on our primal SDP (4).

The Lagrangian dual problem is to maximize this lower bound.

From observation (7), the Lagrangian dual problem is

$$\max_{y \in \mathbb{R}^m, S \in \mathbb{S}_+^n} b^T y$$

subject to

$$\sum_{i=1}^m y_i A_i + S = C$$

$$S \succeq 0$$

(8)

Notice that the dual slack variables $S = C - \sum_{i=1}^m y_i A_i$ constitute a symmetric positive semidefinite matrix in feasible dual solutions.