Integer and Combinatorial Optimization: The Lovasz $\Theta$ Number

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The node packing problem

Definition

A node packing on a graph $G = (V, E)$ is a subset $U \subseteq V$ of the vertices so that no two vertices in $U$ are adjacent. Let $n = |V|$. 

$U = \{1, 5, 6\}$ is a node packing of cardinality 3.
We set up an SDP relaxation for node packing.

We will associate a variable $x_j$ with each vertex.

Note that we must have the product $x_j x_k = 0$ if edge $(j, k)$ is in the graph.

This quadratic representation of the node packing constraint will lead to an SDP formulation.
A scaled incidence vector of a node packing

Say vertices \( \{1, \ldots, p\} \) form a node packing.

We define a scaled incidence vector for this packing:

\[
x_j = \begin{cases} 
\frac{1}{\sqrt{p}} & \text{for } j = 1, \ldots, p \\
0 & \text{for } j = p + 1, \ldots, n
\end{cases}
\]
Constructing a psd matrix $X$

We then construct the matrix $X$ by taking the outer product of vector $x$ with itself:

$$X = xx^T = \begin{bmatrix} 1 & \frac{1}{\sqrt{p}} & \cdots & \frac{1}{\sqrt{p}} \\ \frac{1}{\sqrt{p}} & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{\sqrt{p}} \\ 0 & \cdots & \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{p}} & \cdots & \frac{1}{\sqrt{p}} & 0 & \cdots & 0 \\ \frac{1}{p} & \cdots & \frac{1}{p} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$
There are four crucial observations about the matrix $X$:

- $X$ is symmetric and positive semidefinite.
- The trace of $X$ is equal to 1 since there are $p$ nonzero diagonal entries, each $\frac{1}{p}$.
- The sum of all entries in $X$ is equal to $p$, since there are $p^2$ nonzeroes, each $\frac{1}{p}$.
- $X_{jk} = 0$ if either $x_j = 0$ or $x_k = 0$, that is, if either node $j$ or node $k$ is not in the packing.
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$$\sum x_{jk} = 0 \quad \text{if} \quad \text{edge} \quad (j, k) \in \mathcal{E}.$$
These observations hold for any packing

In particular, for any packing with $p$ vertices, we define

$$x_j = \begin{cases} 
\frac{1}{\sqrt{p}} & \text{if vertex } j \text{ is in the packing} \\
0 & \text{if vertex } j \text{ is not in the packing}
\end{cases}$$

We then construct the matrix $X$ by taking the outer product of vector $x$ with itself. This matrix satisfies our four properties.
An SDP relaxation

This suggests the following SDP relaxation:

\[
\begin{align*}
\max_X & \quad \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} \\
\text{subject to} & \quad \text{trace}(X) = 1 \\
& \quad X_{ij} = 0 \quad \text{if } (i, j) \in E \\
& \quad X \succeq 0
\end{align*}
\]

This is a relaxation: we’ve relaxed the requirement that the rank of \( X \) be equal to 1.

For our example problem, the optimal value to this SDP is equal to 3. There are multiple packings of size 3 so the optimal \( X \) matrix returned by an algorithm is an average of these.
Why does this give a good bound?

Why this relaxation?

We focus on the case where $X$ has rank one, so $X = zz^T$ for some vector $z$.

Since $z_i z_j = X_{ij} = 0$ if $(i, j) \in E$, the support of $z$ must constitute a node packing.

Since $X_{ii} = z_i^2$, the trace constraint requires $z$ to satisfy:

$$\sum_i z_i^2 = 1,$$

so $z$ is a unit vector.
Why does this give a good bound?

The objective function

The objective function can be written in terms of \( z \):

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} Z_i Z_j = \left( \sum_{i=1}^{n} Z_i \right) \left( \sum_{j=1}^{n} Z_j \right) = \left( \sum_{i=1}^{n} Z_i \right)^2.
\]

If \( z \) has a mixture of positive and negative components, we can get a better solution by taking \( |z| \). Thus, we can assume without loss of generality that \( z \geq 0 \), in which case

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} = \|z\|_1^2.
\]

Thus, for a given support of \( z \), we seek to maximize the 1-norm subject to keeping the 2-norm equal to one. The optimal choice is to take every nonzero component of \( z \) to be the same.
Maximizing the one-norm

\[ z_1^2 + z_2^2 = 1 \]

\[ z = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]
The dual SDP

Every SDP has a dual problem.

We’ll give the general form later; for now we write down the dual to the node packing SDP.

Our variables are the scalar variable $\tau$, the symmetric $n \times n$ matrix $y$, and the symmetric $n \times n$ positive semidefinite matrix $S$.

\[
\min_{\tau, y, S} \tau
\subject{to}
\begin{align*}
y_{ij} - S_{ij} &= 1 & \text{for } (i, j) \in E \\
y_{ij} - S_{ij} &= 1 & \text{for } (i, j) \not\in E \\
\tau - S_{ii} &= 1 & \text{for } i = 1, \ldots, n \\
S &\succeq 0
\end{align*}
\]
The clique cover problem

We’ve seen previously that the cardinality of the minimum clique cover and the maximum node packing are the same for perfect graphs.

**Definition**

Given a graph $G = (V, E)$, a **clique cover** of $G$ is a collection of cliques $C_i \subseteq V$ such that $V \subseteq \bigcup_i C_i$, so each vertex in $V$ is in at least one of the cliques.
The example problem

The example problem has a clique cover with 3 cliques:

\[ C_1 = \{1, 4, 7\}, \quad C_2 = \{2, 3, 5\}, \quad C_3 = \{6, 8\}. \]
Constructing a dual SDP solution from a clique cover

Given a clique cover, we can construct a feasible solution to the dual SDP.

The cliques do not have to be maximal, so we can assume the cliques are disjoint: we can shrink them if necessary.

- Let $\tau$ equal the number of cliques.
- Let $y_{ij} = \tau$ if $i$ $j$ are in the same clique.
- Let $y_{ij} = 0$ if $i$ and $j$ are in different cliques.

The equality constraints then determine $S$:

$$S_{ij} = \begin{cases} 
\tau - 1 & \text{if } i = j \\
\tau - 1 & \text{if } i \text{ and } j \text{ in same clique} \\
-1 & \text{if } i \text{ and } j \text{ in different cliques}
\end{cases}$$

Note that the last case includes the case when $(i, j) \notin E$. 
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\[
\begin{align*}
\min & \quad 2 \\
\text{s.t.} & \quad y_{ij} - S_{ij} = 1 \\ & \quad y_{ij} - S_{ij} = 1 \quad (i, j) \in E \\
& \quad -S_{ij} = 1 \quad (i, j) \notin E \\
& \quad \tau - 2 - S_{ij} = 1 \\
& \quad \forall : \quad S_{ij} \geq 0
\end{align*}
\]
The dual slack matrix $S$ for our example

For the example problem with the clique cover $C_1 = \{1, 4, 7\}$, $C_2 = \{2, 3, 5\}$, $C_3 = \{6, 8\}$, we get $\tau = 3$ and so

$$S = \begin{bmatrix}
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\
-1 & 2 & 2 & -1 & 2 & -1 & -1 & -1 \\
-1 & 2 & 2 & -1 & 2 & -1 & -1 & -1 \\
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\
-1 & 2 & 2 & -1 & 2 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 \\
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\
-1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 \\
\end{bmatrix}$$

We need to show that this matrix is positive semidefinite.
Expressing $S$ as a sum of psd matrices

$$S = \begin{bmatrix}
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\
-1 & 2 & 2 & -1 & 2 & -1 & -1 & -1 \\
-1 & 2 & 2 & -1 & 2 & -1 & -1 & -1 \\
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\
-1 & 2 & 2 & -1 & 2 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 \\
-1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 \\
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\
-1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 \\
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
-1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
\end{bmatrix}$$

$$= \begin{bmatrix}
\{1,4,7,\} & \{6,8\} \\
\{2,3,3\} & \{4,6,8\} \\
\{1,4,7\} & \{2,3,5\} \\
\end{bmatrix} + \begin{bmatrix}
1 & -1 & -1 & 1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 1 & -1 & 1 & 0 & -1 & 0 \\
-1 & 1 & 1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & -1 & 1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 1 & -1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$
Getting outer products

Note that

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
-1 & 0 & 0 & -1 & 0 & 1 & -1 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-1 \\
0 \\
0 \\
-1 \\
0 \\
1 \\
1 \\
-1 \\
1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 & -1 & 0 & 1 & -1 & 1
\end{bmatrix}
\]

A similar relationship holds for the other matrices.
The structure of the dual slack matrix

Note the pattern:

each of the three matrices in the sum picks two of the three cliques and sets $s_j = 1$ for the vertices in one clique and $s_j = -1$ for vertices in the other clique, with the matrix equal to the outer product $ss^T$.

This construction holds in general.

If we have $q$ cliques then the matrix $S$ is a sum of $\binom{q}{2}$ rank one matrices, each formed from picking two of the cliques, constructing a vector $s$, and taking the outer product $ss^T$. 
Solving the SDP to optimality

In general, semidefinite programs may have a duality gap, with the primal and dual optimal values different from each other.

In the node packing construction, there is no duality gap, because there is a dual feasible solution with a positive definite $S$:

$\text{take } \tau > n \text{ and all } y_{ij} = 0, \text{ and then } S \text{ is diagonally dominant, with } S_{ii} > n - 1 \text{ and all off-diagonal } S_{ij} = -1.$

Semidefinite programs can be solved in polynomial time.

Consequently, the node packing problem on perfect graphs can be solved in polynomial time by solving its SDP formulation, since there is no SDP duality gap and the dual solution is the value of the best clique cover, which is equal to the value of the maximum node packing for a perfect graph.