Integer and Combinatorial Optimization: Branch-and-Cut

John E. Mitchell

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

November 2018
Outline

1. Cutting planes
2. The branch-and-cut algorithm
3. Example
A cutting plane example

Initial relaxation:

Solve linear optimization relaxation

\[
\min \{ c^T x : Ax \leq b, x \geq 0 \}
\]
A cutting plane example

Initial relaxation:

Solve linear optimization relaxation

\[
\min \{c^T x : Ax \leq b, x \geq 0\}
\]
Add a cutting plane and reoptimize

Solve linear optimization relaxation

$$\min \{ c^T x : Ax \leq b, \ x \geq 0, \ a_1^T x \leq b_1 \}$$

Cutting plane $a_1^T x = b_1$
Add a cutting plane and reoptimize

Solve linear optimization relaxation

$$\min \{ c^T x : Ax \leq b, \ x \geq 0, \ a_1^T x \leq b_1 \}$$

Cutting plane $$a_1^T x = b_1$$

Cutting plane $$a_2^T x = b_2$$

Solution to linear optimization relaxation
Add a cutting plane and reoptimize

Solve linear optimization relaxation

\[ \min \{ c^T x : Ax \leq b, x \geq 0, a_1^T x \leq b_1, a_2^T x \leq b_2 \} \]

Solution to linear optimization relaxation

- Cutting plane \( a_1^T x = b_1 \)
- Cutting plane \( a_2^T x = b_2 \)

Solution to relaxation is integral, so it solves the integer optimization problem.
Outline

1. Cutting planes
2. The branch-and-cut algorithm
3. Example
Branch-and-cut

Branch-and-bound can be improved through the addition of cutting planes. We work with integer optimization problems of the form

$$\min \ c^T x$$
$$\text{subject to } A x \leq b$$
$$x \geq 0, \ x \in \mathbb{R}^n$$
$$x_i \text{ integer, } i = 1, \ldots, p.$$
**The steps of the algorithm**

1. **Initialize:** The initial set $L$ of active nodes consists of just one problem, $L = \{(IOP)\}$. If a feasible solution $\bar{x}$ is known, the initial upper bound on the optimal value of $(IOP)$ is set to $z^u = c^T \bar{x}$; else, we initialize $z^u = \infty$.

2. **Termination:** If $L = \emptyset$ then the feasible integral point that provided the incumbent upper bound $z^u$ is optimal for $(IOP)$.

3. **Relaxation:** Remove a node $\text{IOP}^l$ from the set of active nodes. Solve the linear optimization relaxation of $\text{IOP}^l$. Let $z^l$ be the optimal value of the relaxation (we allow $z^l = \infty$ if the relaxation is infeasible, and $z^l = -\infty$ if the relaxation is unbounded). Let $x^l$ be an optimal solution to this relaxation, if the relaxation has an optimal solution. If the relaxation has an unbounded optimal value then let $x^l$ either be a fractional extreme ray or a feasible point for the relaxation with value smaller than $z^u$. 
**The steps of the algorithm**

1. **Initialize:** The initial set $L$ of active nodes consists of just one problem, $L = \{(IOP)\}$. If a feasible solution $\bar{x}$ is known, the initial upper bound on the optimal value of $(IOP)$ is set to $z^u = c^T \bar{x}$; else, we initialize $z^u = \infty$.

2. **Termination:** If $L = \emptyset$ then the feasible integral point that provided the incumbent upper bound $z^u$ is optimal for $(IOP)$.

3. **Relaxation:** Remove a node $IOP^l$ from the set of active nodes. Solve the linear optimization relaxation of $IOP^l$. Let $z^l$ be the optimal value of the relaxation (we allow $z^l = \infty$ if the relaxation is infeasible, and $z^l = -\infty$ if the relaxation is unbounded). Let $x^l$ be an optimal solution to this relaxation, if the relaxation has an optimal solution. If the relaxation has an unbounded optimal value then let $x^l$ either be a fractional extreme ray or a feasible point for the relaxation with value smaller than $z^u$. 

[Mitchell Branch-and-Cut 8 / 21]
The branch-and-cut algorithm

The steps of the algorithm

1. **Initialize:** The initial set $L$ of active nodes consists of just one problem, $L = \{(IOP)\}$. If a feasible solution $\bar{x}$ is known, the initial upper bound on the optimal value of $(IOP)$ is set to $z^u = c^T \bar{x}$; else, we initialize $z^u = \infty$.

2. **Termination:** If $L = \emptyset$ then the feasible integral point that provided the incumbent upper bound $z^u$ is optimal for $(IOP)$.

3. **Relaxation:** Remove a node $IOP^l$ from the set of active nodes. Solve the linear optimization relaxation of $IOP^l$. Let $z^l$ be the optimal value of the relaxation (we allow $z^l = \infty$ if the relaxation is infeasible, and $z^l = -\infty$ if the relaxation is unbounded). Let $x^l$ be an optimal solution to this relaxation, if the relaxation has an optimal solution. If the relaxation has an unbounded optimal value then let $x^l$ either be a fractional extreme ray or a feasible point for the relaxation with value smaller than $z^u$. 

Mitchell  
Branch-and-Cut  
8 / 21
The branch-and-cut algorithm

The steps (continued)

4. **Add cutting planes:** If desired, search for cutting planes that are violated by $x^l$; if any are found, add them to the relaxation and return to Step 2.

5. **Fathom by infeasibility:** If the relaxation is infeasible then $IOP^l$ is fathomed. Return to Step 2.

6. **Fathom by integrality:** If $x^l$ is integral then $IOP^l$ is fathomed. Update $z^u \leftarrow \min \{c^T x^l, z^u\}$. Return to Step 2.

7. **Fathom by bounds:** If $c^T x^l \geq z^u$ then $IOP^l$ is fathomed. Return to Step 2.

8. **Subdivide:** Choose a component $i$ with $x^l_i$ fractional. Create two new nodes and add them to the set of active nodes:
   (a) $x$ feasible in $(IOP^l)$ with $x_i \leq \lfloor x^l_i \rfloor$, and
   (b) $x$ feasible in $(IOP^l)$ with $x_i \geq \lceil x^l_i \rceil$.
   Return to Step 2.
The branch-and-cut algorithm

The steps (continued)

4 **Add cutting planes:** If desired, search for cutting planes that are violated by \( x^l \); if any are found, add them to the relaxation and return to Step 2.

5 **Fathom by infeasibility:** If the relaxation is infeasible then \( IOP^l \) is fathomed. Return to Step 2.

6 **Fathom by integrality:** If \( x^l \) is integral then \( IOP^l \) is fathomed. Update \( z^u \leftarrow \min\{c^T x^l, z^u\} \). Return to Step 2.

7 **Fathom by bounds:** If \( c^T x^l \geq z^u \) then \( IOP^l \) is fathomed. Return to Step 2.

8 **Subdivide:** Choose a component \( i \) with \( x_i^l \) fractional. Create two new nodes and add them to the set of active nodes:
   (a) \( x \) feasible in \( (IOP^l) \) with \( x_i \leq \lfloor x_i^l \rfloor \), and
   (b) \( x \) feasible in \( (IOP^l) \) with \( x_i \geq \lceil x_i^l \rceil \).
Return to Step 2.
The branch-and-cut algorithm

The steps (continued)

4 **Add cutting planes:** If desired, search for cutting planes that are violated by \( x^l \); if any are found, add them to the relaxation and return to Step 2.

5 **Fathom by infeasibility:** If the relaxation is infeasible then \( IOP^l \) is fathomed. Return to Step 2.

6 **Fathom by integrality:** If \( x^l \) is integral then \( IOP^l \) is fathomed. Update \( z^u \leftarrow \min\{c^T x^l, z^u\} \). Return to Step 2.

7 **Fathom by bounds:** If \( c^T x^l \geq z^u \) then \( IOP^l \) is fathomed. Return to Step 2.

8 **Subdivide:** Choose a component \( i \) with \( x^l_i \) fractional. Create two new nodes and add them to the set of active nodes:
   (a) \( x \) feasible in \( (IOP^l) \) with \( x_i \leq \lfloor x^l_i \rfloor \), and
   (b) \( x \) feasible in \( (IOP^l) \) with \( x_i \geq \lceil x^l_i \rceil \).
   Return to Step 2.
The branch-and-cut algorithm

The steps (continued)

4. **Add cutting planes**: If desired, search for cutting planes that are violated by $x^l$; if any are found, add them to the relaxation and return to Step 2.

5. **Fathom by infeasibility**: If the relaxation is infeasible then $IOP^l$ is fathomed. Return to Step 2.

6. **Fathom by integrality**: If $x^l$ is integral then $IOP^l$ is fathomed. Update $z^u \leftarrow \min\{c^T x^l, z^u\}$. Return to Step 2.

7. **Fathom by bounds**: If $c^T x^l \geq z^u$ then $IOP^l$ is fathomed. Return to Step 2.

8. **Subdivide**: Choose a component $i$ with $x_i^l$ fractional. Create two new nodes and add them to the set of active nodes:
   (a) $x$ feasible in $(IOP^l)$ with $x_i \leq \lfloor x_i^l \rfloor$, and
   (b) $x$ feasible in $(IOP^l)$ with $x_i \geq \lceil x_i^l \rceil$.
   Return to Step 2.
4. **Add cutting planes:** If desired, search for cutting planes that are violated by $x^l$; if any are found, add them to the relaxation and return to Step 2.

5. **Fathom by infeasibility:** If the relaxation is infeasible then $IOP^l$ is fathomed. Return to Step 2.

6. **Fathom by integrality:** If $x^l$ is integral then $IOP^l$ is fathomed. Update $z^u \leftarrow \min\{c^T x^l, z^u\}$. Return to Step 2.

7. **Fathom by bounds:** If $c^T x^l \geq z^u$ then $IOP^l$ is fathomed. Return to Step 2.

8. **Subdivide:** Choose a component $i$ with $x_i^l$ fractional. Create two new nodes and add them to the set of active nodes:
   
   (a) $x$ feasible in ($IOP^l$) with $x_i \leq \lfloor x_i^l \rfloor$, and
   
   (b) $x$ feasible in ($IOP^l$) with $x_i \geq \lceil x_i^l \rceil$.

   Return to Step 2.
Notes on branch-and-cut

$L$ is the set of active nodes in the branch-and-cut tree. The value of the best known feasible point for \((IOP)\) is $z^u$, which provides an upper bound on the optimal value of \((IOP)\).

In some situations, a very large number of violated cutting planes are found in Step 4, in which case it is common to sort the cutting planes somehow (perhaps by violation), and add just a subset.
Families of cutting planes

Many structured integer optimization problems have known families of valid inequalities. For example, the subtour elimination constraints are valid constraints for the traveling salesman problem.

Use subtour elimination constraint to tighten relaxation:

\[ x_{rs} + x_{st} + x_{rt} \leq 2 \]
Solving the relaxations

The relaxations can be solved using any method for linear programming problems.

Typically, the initial relaxation is solved using the simplex method.

Subsequent relaxations are solved using the dual simplex method, since the dual solution for the relaxation of the parent subproblem is still feasible in the relaxation of the child subproblem.

Further, when cutting planes are added in Step 4, the current iterate is still dual feasible, so again the modified relaxation can be solved using the dual simplex method.

It is also possible to use an interior point method, and this can be a good choice if the linear programming relaxations are large.
Further notes

Of course, there are several issues to be resolved with this algorithm. These include the major questions of deciding whether to branch or to cut and deciding how to branch and how to generate cutting planes.

If the objective function and/or the constraints in \( IOP \) are nonlinear, the problem can still be attacked with a branch-and-cut approach.
Outline

1. Cutting planes

2. The branch-and-cut algorithm

3. Example
A branch-and-cut example

The integer programming problem

\[
\begin{align*}
\min \quad & z := -6x_1 - 5x_2 \\
\text{subject to} \quad & 3x_1 + x_2 \leq 11 \quad (Eg0) \\
& -x_1 + 2x_2 \leq 5 \\
& x_1, x_2 \geq 0, \text{ integer.}
\end{align*}
\]

is illustrated in the figure on the next slide. The feasible integer points are marked. The \emph{linear optimization relaxation} (or \emph{LOP relaxation}) is obtained by ignoring the integrality restrictions and is indicated by the polyhedron contained in the solid lines.
Solve the initial relaxation \((Eg0)\)

Optimal solution to initial relaxation is \(x = (2 \frac{3}{7}, 3 \frac{5}{7})\), with value \(-33\frac{1}{7}\). Fractional, so branch on \(x_1 \leq 2\) versus \(x_1 \geq 3\).
Solve the initial relaxation \((Eg0)\)

Optimal solution to initial relaxation is \(x = (2\frac{3}{7}, 3\frac{5}{7})\), with value \(-33\frac{1}{7}\). Fractional, so branch on \(x_1 \leq 2\) versus \(x_1 \geq 3\).
Solve the initial relaxation \((Eg0)\)

Optimal solution to initial relaxation is \(x = (2\frac{3}{7}, 3\frac{5}{7})\), with value \(-33\frac{1}{7}\). Fractional, so branch on \(x_1 \leq 2\) versus \(x_1 \geq 3\).
Branch with $x_1 \geq 3$

$$\begin{align*}
\text{min } & \quad z := -6x_1 - 5x_2 \\
\text{s.t. } & \quad 3x_1 + x_2 \leq 11 \\
& \quad -x_1 + 2x_2 \leq 5 \quad (Eg1) \\
& \quad x_1 \geq 3 \\
& \quad x_1, x_2 \geq 0, \text{ integer.}
\end{align*}$$

Solution: $x = (3, 2)$. Integral, so get new incumbent solution, new upper bound on optimal value of -28.
Branch with $x_1 \geq 3$

$$
\begin{align*}
\min & \quad z := -6x_1 - 5x_2 \\
\text{s.t.} & \quad 3x_1 + x_2 \leq 11 \\
& \quad -x_1 + 2x_2 \leq 5 \quad (Eg1) \\
& \quad x_1 \geq 3 \\
& \quad x_1, x_2 \geq 0, \text{ integer.}
\end{align*}
$$

Solution: $x = (3, 2)$.
Integral, so get new incumbent solution, new upper bound on optimal value of -28.
Branch with \( x_1 \geq 3 \)

\[
\begin{align*}
\text{min} \quad & z := -6x_1 - 5x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \leq 11 \\
& -x_1 + 2x_2 \leq 5 \quad (Eg1) \\
& x_1 \geq 3 \\
& x_1, x_2 \geq 0, \text{ integer.}
\end{align*}
\]

Solution: \( x = (3, 2) \).
Integral, so get new incumbent solution, new upper bound on optimal value of -28.
Branch with $x_1 \geq 3$

\[
\begin{align*}
\text{min } & \quad z := -6x_1 - 5x_2 \\
\text{s.t.} & \quad 3x_1 + x_2 \leq 11 \\
& \quad -x_1 + 2x_2 \leq 5 \quad (Eg1) \\
& \quad x_1 \geq 3 \\
& \quad x_1, x_2 \geq 0, \text{ integer.}
\end{align*}
\]

Solution: $x = (3, 2)$. Integral, so get new incumbent solution, new upper bound on optimal value of -28.
Branch with $x_1 \leq 2$

$$\begin{align*}
\text{min} \quad & z := -6x_1 - 5x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \leq 11 \\
& -x_1 + 2x_2 \leq 5 \quad (Eg2) \\
& x_1 \leq 2 \\
& x_1, x_2 \geq 0, \text{ integer.}
\end{align*}$$

Solution: $x = (2, 3.5)$, value -29.5.
Fractional, so add cutting plane

$2x_1 + x_2 \leq 7.$
Branch with $x_1 \leq 2$

\[
\begin{align*}
\min \quad & z := -6x_1 - 5x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \leq 11 \\
& -x_1 + 2x_2 \leq 5 \quad (Eg2) \\
& x_1 \leq 2 \\
& x_1, x_2 \geq 0, \text{ integer.}
\end{align*}
\]

Solution: $x = (2, 3.5)$, value -29.5. Fractional, so add cutting plane $2x_1 + x_2 \leq 7$. 
Branch with \( x_1 \leq 2 \)

\[
\begin{align*}
\text{min} & \quad z := -6x_1 - 5x_2 \\
\text{s.t.} & \quad 3x_1 + x_2 \leq 11 \\
& \quad -x_1 + 2x_2 \leq 5 \quad (Eg2) \\
& \quad x_1 \leq 2 \\
& \quad x_1, x_2 \geq 0, \text{ integer.}
\end{align*}
\]

Solution: \( x = (2, 3.5) \), value \(-29.5\).
Fractional, so add cutting plane \( 2x_1 + x_2 \leq 7 \).
Branch with $x_1 \leq 2$

$$\begin{align*}
\min \quad & z := -6x_1 - 5x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \leq 11 \\
& -x_1 + 2x_2 \leq 5 \quad (Eg2) \\
& x_1 \leq 2 \\
& x_1, x_2 \geq 0, \text{ integer.}
\end{align*}$$

Solution: $x = (2, 3.5)$, value -29.5. Fractional, so add cutting plane $2x_1 + x_2 \leq 7$. 
Add cutting plane $2x_1 + x_2 \geq 7$

$$\begin{align*}
\text{min } z &= -6x_1 - 5x_2 \\
\text{s.t. } 3x_1 + x_2 &\leq 11 \\
-x_1 + 2x_2 &\leq 5 \quad (Eg3) \\
x_1 &\leq 2 \\
2x_1 + x_2 &\leq 7 \\
x_1, x_2 &\geq 0, \text{ integer.}
\end{align*}$$

Solution: $x = (1.8, 3.4)$, with value $-27.8$. Value worse than incumbent integer solution, so fathomed by bounds.
Add cutting plane $2x_1 + x_2 \geq 7$

$$
\begin{align*}
\text{min} & \quad z := -6x_1 - 5x_2 \\
\text{s.t.} & \quad 3x_1 + x_2 \leq 11 \\
& \quad -x_1 + 2x_2 \leq 5 \quad (Eg3) \\
& \quad x_1 \leq 2 \\
& \quad 2x_1 + x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0, \ \text{integer}.
\end{align*}
$$

Solution: $x = (1.8, 3.4)$, with value $-27.8$. Value worse than incumbent integer solution, so fathomed by bounds.
Add cutting plane $2x_1 + x_2 \geq 7$

\[
\begin{align*}
\text{min} & \quad z := -6x_1 - 5x_2 \\
\text{s.t.} & \quad 3x_1 + x_2 \leq 11 \\
& \quad -x_1 + 2x_2 \leq 5 \quad (Eg3) \\
& \quad x_1 \leq 2 \\
& \quad 2x_1 + x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0, \text{ integer.}
\end{align*}
\]

Solution: $x = (1.8, 3.4)$, with value $-27.8$. Value worse than incumbent integer solution, so fathomed by bounds.
Add cutting plane $2x_1 + x_2 \geq 7$

\[
\begin{align*}
\text{min} \quad & z := -6x_1 - 5x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \leq 11 \\
& -x_1 + 2x_2 \leq 5 \quad (Eg3) \\
& x_1 \leq 2 \\
& 2x_1 + x_2 \leq 7 \\
& x_1, x_2 \geq 0, \text{ integer.}
\end{align*}
\]

Solution: $x = (1.8, 3.4)$, with value $-27.8$. Value worse than incumbent integer solution, so fathomed by bounds.
Lifting

Notice that the cutting plane introduced in the second subproblem is not valid for the first subproblem.

This inequality can be modified to make it valid for the first subproblem by using a *lifting* technique.

(Not discussed further in this course.)
The branch-and-cut tree
The progress of the algorithm is illustrated below.

(Eg0). Value $-33\frac{1}{7}$
Solution $x = (2\frac{3}{7}, 3\frac{5}{7})$

$\begin{align*}
x_1 &\leq 2 \\
x_1 &\geq 3
\end{align*}$

(Eg1). Value $-28$
Solution $x = (3, 2)$

(Eg2). Value $-29.5$
Solution $x = (2, 3.5)$

$\begin{align*}
2x_1 + x_2 &\leq 7
\end{align*}$

(Eg3). Value $-27.8$
Solution $x = (1.8, 3.4)$