Integer and Combinatorial Optimization: Branch-and-Bound

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Outline

1. Branch-and-Bound

2. Statement of the algorithm

3. An example
Integer optimization problems

We have an integer optimization problem

\[
\min_{x \in \mathbb{R}^n} \quad c^T x \\
\text{subject to} \quad Ax = b \quad \quad (IOP) \\
x \geq 0 \\
x \text{ integer}
\]

To simplify the presentation, we’ve constrained all the variables to be integral. The branch-and-bound algorithm can be readily generalized to \textit{mixed} integer optimization problems where only some of the variables are constrained to be integral.
Linear optimization relaxation

We have a linear optimization relaxation obtained by dropping the integrality restriction:

$$\min_{x \in \mathbb{R}^n} c^T x$$
subject to
$$Ax = b$$
$$x \geq 0$$

The feasible region of the relaxation is denoted $F$, so

$$F = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}.$$

The progress of branch-and-bound can be regarded as a tree: we start off with a root problem, which we repeatedly subdivide. Each problem is called a node of the tree. Each subdivision can be pictured as branches coming off the current node, leading to two or more child nodes.
A branch-and-bound tree

Root node:
\[ \min \{ c^T x : x \in F \} \]

\[ x_2 \leq 1 \]

\[ x_3 \leq 3 \]

\[ x_3 \geq 4 \]

\[ x_2 \geq 2 \]
Fathoming

We do not need to branch further at a particular node if one of three things happens:

- The relaxation is **infeasible**, in which case the underlying integer optimization problem at that node is also infeasible.
- The optimal solution to the relaxation is **integral**, in which case this solution also solves the underlying integer optimization problem at that node.
- The optimal value of the relaxation is **at least as large** as that of the best known integer feasible solution to the original problem. In this case, no integer feasible solution at the node can be better than this already-known solution.
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Active nodes

The algorithm maintains a list of active nodes that have not yet been fathomed and are candidates for branching.

At each iteration, a node is selected from the list, its relaxation is solved, and the node is either fathomed or it is subdivided creating two new active nodes.
The algorithm

1. **Initialize:** The initial set $L$ of active nodes consists of just one problem, $L = \{(IOP)\}$. If a feasible solution $\bar{x}$ is known, the initial upper bound on the optimal value of $(IOP)$ is set to $z^u = c^T \bar{x}$; else, we initialize $z^u = \infty$.

2. **Termination:** If $L = \emptyset$ then the feasible integral point that provided the incumbent upper bound $z^u$ is optimal for $(IOP)$.

3. **Relaxation:** Remove a node $IOP^l$ from the set of active nodes. Solve the linear optimization relaxation of $IOP^l$. Let $z^l$ be the optimal value of the relaxation (we allow $z^l = \infty$ if the relaxation is infeasible, and $z^l = -\infty$ if the relaxation is unbounded). Let $x^l$ be an optimal solution to this relaxation, if the relaxation has an optimal solution. If the relaxation has an unbounded optimal value then let $x^l$ either be a fractional extreme ray or a feasible point for the relaxation with value smaller than $z^u$.
The algorithm

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The algorithm

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The algorithm

4 **Fathom by infeasibility:** If the relaxation is infeasible then \( IOP^l \) is fathomed. Return to Step 2.

5 **Fathom by integrality:** If \( x^l \) is integral then \( IOP^l \) is fathomed. Update \( z^u \leftarrow \min \{c^T x^l, z^u\} \). Return to Step 2.

6 **Fathom by bounds:** If \( c^T x^l \geq z^u \) then \( IOP^l \) is fathomed. Return to Step 2.

7 **Subdivide:** Choose a component \( i \) with \( x^l_i \) fractional. Create two new nodes and add them to the set of active nodes:
   (a) \( x \) feasible in \( (IOP^l) \) with \( x_i \leq \lfloor x^l_i \rfloor \), and
   (b) \( x \) feasible in \( (IOP^l) \) with \( x_i \geq \lceil x^l_i \rceil \).
   Return to Step 2.
The algorithm

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7 **Subdivide:** Choose a component $i$ with $x_i^l$ fractional. Create two new nodes and add them to the set of active nodes:
   (a) $x$ feasible in $(IOP^l)$ with $x_i \leq \lfloor x_i^l \rfloor$, and
   (b) $x$ feasible in $(IOP^l)$ with $x_i \geq \lceil x_i^l \rceil$.
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   Return to Step 2.
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An example

We look at the integer optimization problem

$$\min_{x \in \mathbb{R}^3} \quad 5x_1 + 5x_2 - 13x_3$$
subject to

$$x_1 + x_2 + x_3 \leq 6$$
$$10x_1 - 8x_2 \leq 15$$
$$-6x_1 - x_2 + 9x_3 \leq 9$$

$$x_i \geq 0, \text{ } x_i \text{ integer, } i = 1, 2, 3$$

(IOP)

It’s easy to find a feasible solution for this problem: $x = (0, 0, 0)$. Hence we can initialize with $z^u = 0$.

We let

$$F = \{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 \leq 6, 10x_1 - 8x_2 \leq 15, -6x_1 - x_2 + 9x_3 \leq 9, x_1, x_2, x_3 \geq 0 \}.$$
The tree for the example

Root node:
\[
\begin{align*}
\min \{ c^T x : & \quad x \in F \\
x^0 = (1.5, 0, 2), & \\
z^0 = -18.5
\end{align*}
\]
The tree for the example

Root node:
\[ \min \{ c^T x : \quad x \in F \} \]
\[ x^0 = (1.5, 0, 2), \quad z^0 = -18.5 \]

\[ x_1 \leq 1 \]

\[ x_1 \geq 2 \]

\[ x^2 = (2, \frac{5}{8}, 2 \frac{29}{72}), \quad z^2 = -18 \frac{1}{9} \]
The tree for the example

Root node:
\[
\min \{ c^T x : x \in F \}
\]

\[
x^0 = (1.5, 0, 2),
\]

\[
z^0 = -18.5
\]

\[
x_1 \leq 1
\]

\[
x_1 \geq 2
\]

\[
x^1 = (1, 0, 1 \frac{2}{3}),
\]

\[
z^1 = -16 \frac{2}{3}
\]

\[
x^2 = (2, \frac{5}{8}, 2 \frac{29}{72}),
\]

\[
z^2 = -18 \frac{1}{9}
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The tree for the example

Root node:
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x^1 = (1, 0, 1\frac{2}{3}), \quad z^1 = -16\frac{2}{3}
\]

\[
x^2 = (2, 2\frac{5}{8}, 2\frac{29}{72}), \quad z^2 = -18\frac{1}{9}
\]

\[
x_1 \geq 2
\]

\[
x_3 \geq 3
\]

infeasible,
\[
z^6 = +\infty
\]

fathom by infeasibility
An example

The tree for the example

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x_0 = (1.5, 0, 2),
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\]
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x_1 \geq 2
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x_1 \leq 1
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x^1 = (1, 0, 1\frac{2}{3}),
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x^2 = (2, \frac{5}{8}, 2\frac{29}{72}),
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\]
\[
x_3 \geq 3
\]
\[
x_3 \leq 2
\]
\[
x^5 = (2, \frac{5}{8}, 2),
\]
\[
z^5 = -12\frac{7}{8}
\]
\[
infeasible,
\]
\[
z^6 = +\infty
\]
\[
fathom by infeasibility
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The tree for the example

Root node:
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\min \{ c^T x : x \in F \}
\]
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x^0 = (1.5, 0, 2),
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x^1 = (1, 0, 1\frac{2}{3}),
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\[
x_1 \leq 1
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x^2 = (2, \frac{5}{8}, 2\frac{29}{72}),
\]
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z^2 = -18\frac{1}{9}
\]

\[
x_3 \geq 3
\]

\[
x_3 \leq 2
\]

\[
x^3 = (0, 0, 1),
\]
\[
z^3 = -13
\]

fathom by integrality

\[
x^5 = (2, \frac{5}{8}, 2),
\]
\[
z^5 = -12\frac{7}{8}
\]

fathom by bounds

\[
x^6 = +\infty
\]

infeasible, fathom by infeasibility
The tree for the example

Root node:
\[
\begin{align*}
\min \{ c^T x : & \quad x \in F \} \\
x^0 = (1.5, 0, 2), & \quad z^0 = -18.5
\end{align*}
\]

\[x_1 \geq 2\]

\[x_1 \leq 1\]

\[x_3 \geq 2\]

\[x_3 \leq 3\]

\[x_3 \leq 2\]

\[x_3 \geq 3\]

\[x^1 = (1.0, 1.2/3), \quad z^1 = -16.2/3\]

\[x^2 = (2.5/8, 2.29/72), \quad z^2 = -18.1/9\]

\[x^3 = (0, 0, 1), \quad z^3 = -13 \quad \text{fathom by integrality}\]

\[x^4 = (1, 3, 2), \quad z^4 = -6 \quad \text{fathom by bounds}\]

\[x^5 = (2, 5/8, 2), \quad z^5 = -127/8 \quad \text{fathom by bounds}\]

infeasible, \[z^6 = +\infty \quad \text{fathom by infeasibility}\]
The tree for the example

Root node: 
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\begin{align*}
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x^1 = (1, 0, 1\frac{2}{3}), \\
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x^3 = (0, 0, 1), \\
z^3 = -13
\]

fathom by integrality

\[
x^4 = (1, 3, 2), \\
z^4 = -6
\]

fathom by bounds

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x^5 = (2, \frac{5}{8}, 2), \\
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fathom by bounds

infeasible, 
\[
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fathom by infeasibility

The optimal solution to \((IOP)\) is \(x^* = (0, 0, 1)\), with value \(z^* = -13\).
Choosing the branching variable

Many rules have been proposed for making choices in branch-and-bound. In the worst case, the algorithm can generated an exponential number of nodes. So heuristics are needed to try to shrink the number of explored nodes.

Performing some sort of look-ahead appears to be the best way to manage the size of the tree: we examine several different choices for branching variable and choose the one that gives the children with the best bounds. However, this requires a lot of work at each node. One variant of this approach is known as strong branching.

Another approach is to try to learn from earlier branching decisions: how much progress did we make when we branched on \( x_7 \) before? This leads to something called pseudocosts, which are estimates of the progress we could make.
Choosing the next node

In the formal statement of the algorithm, we construct the child nodes as soon as we determine the node cannot be immediately fathomed. It may save time to not make this decision immediately, but to \textit{wait}: it may be possible to fathom this node later by bounds, if another good feasible solution arises. Formally, we could define another set of nodes whose relaxations have been solved but which are neither fathomed nor yet subdivided.

A common choice for the next node is to choose the one whose \textit{parent has the smallest value}. The reason is that we will have to examine this node at some point: it can never be fathomed by bounds.