Integer and Combinatorial Optimization: Perfect Graphs

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Perfect graphs

See Nemhauser and Wolsey, section III.1.5, for more information.

Definition

A graph $G = (V, E)$ is perfect if $\chi(\hat{G}) = \omega(\hat{G})$ for every induced subgraph $\hat{G}$, where

$\chi(G) = \text{chromatic number of } G$

$= \text{least number of colors needed to color the vertices so no two adjacent vertices have the same color}$

$\omega(G) = \text{size of maximum clique of } G$

Note that every graph has $\chi(G) \geq \omega(G)$, since every vertex in the largest clique requires a different color.
\[ \omega(\phi) = 3 \]

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Bipartite graphs

Example

Bipartite graphs are perfect:

\[ \chi(G) = \omega(G) = 2 \]
Complements of bipartite graphs

Example

Complements of bipartite graphs are perfect:

\[ \chi(G) = \omega(G) = 3 \]
$\chi(G) = \omega(G) = 2$.

$\chi(\overline{G}) = \omega(\overline{G}) = 4$.
Odd holes

Example

Odd holes are not perfect:

\(\chi(G) = 3, \omega(G) = 2\)
Odd anitholes

Example

Odd anitholes are not perfect:

\[ \chi(G) = 4, \quad \omega(G) = 3 \]
Complement

also a cycle of length 5.
Complements of perfect graphs

A graph is perfect if and only if its complement is perfect.

A clique in a graph corresponds to a node packing in its complement.

A coloring in a graph corresponds to a clique cover in its complement; that is, a collection of cliques so that each vertex is in (at least) one clique.
Given alternative characterization of perfect graphs

clique

$G$ clique $\iff$ node packing $\overline{G}$ clique cover: divide vertices into cliques.
A packing LP

Consider the packing linear program:

\[ \max_{x \in \mathbb{R}^n} c^T x \]
\[ \text{subject to} \quad Ax \leq e \]
\[ x \geq 0 \]

where \( e \) is the vector of ones, and every entry in \( A \) is either 0 or 1. We construct a graph:

- Each column of \( A \) gives a node in a graph.
- Each row of \( A \) gives a clique in the graph. So if \( a_{ij} = a_{ik} = 1 \) then vertices \( j \) and \( k \) are adjacent.
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Constructing a graph from a matrix

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For example:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ r & s & t & u & v \end{bmatrix}$$

$\chi(G) = \omega(G) = 3$

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0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\[
A_x = \begin{bmatrix}
 x_r \\
 x_s \\
 x_t \\
 x_u \\
 x_v \\
\end{bmatrix} \leq 1
\]

\[
\chi(G) = \omega(G) = 3 \\
\chi(\bar{G}) = \omega(\bar{G}) = 2
\]
Complement:

Parking: \{ s, u \}

Clique cover:
\{ \{ r, s, k \}, \{ u, v \} \}

\( |\{s\}| + |\{u\}| = 2 \)

Clique: \{ s, u \}
Binary solutions to the packing LP correspond to *node packings in the original graph* and to *cliques in the complement of the graph*:

in a clique $C$ in the complement, none of the vertices are adjacent in the original graph, so there is no clique in the original graph that contains two of the vertices in $C$. 
Clique covers

Binary **dual** solutions to the packing LP with \( c = e \) correspond to **coverings by cliques in the original graph** and to **colorings in the complement of the graph**.

The dual problem is

\[
\begin{align*}
\min_{x \in \mathbb{R}^m} & \quad e^T y \\
\text{subject to} & \quad A^T y \geq e \\
& \quad y \geq 0
\end{align*}
\]

which is a covering problem.

Every vertex must be covered by a color.

If two vertices are not adjacent they must have different colors; this corresponds to the two columns not appearing in the same clique in the original graph.
Two theorems

**Theorem (Chvatal)**

The LP has an integral optimal solution for every nonnegative integral vector $c$ if and only if the underlying graph is perfect.

**Theorem**

(Perfect graph conjecture of Berge (1961). Proved by Chudnovsky, Robertson, Seymour, and Thomas (2003).) A graph is perfect if and only if it contains no odd holes and no odd antiholes.
Two theorems

Theorem (Chvatal)

The LP has an integral optimal solution for every nonnegative integral vector $c$ if and only if the underlying graph is perfect.

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The Petersen graph

Example

\[ \chi(\bar{G}) = 5. \text{ E.g. clique cover } \{r, w\}, \{s, p\}, \{t, q\}, \{u, k\}, \{v, l\} \]

\[ \omega(\bar{G}) = 4. \text{ E.g. packing } \{r, t, l, k\} \]
Look for packings.
Have 2 cycles of length 5.
So at most 4 vertices in a packing.
Another example

Example

A graph that is not perfect because an induced subgraph is not perfect:

Here we have $\chi(\bar{G}) = \omega(\bar{G}) = 3$. However, if we restrict attention to the subset $\{r, s, t, u, v\}$ of the vertices then this subgraph is an odd hole, so this is not perfect. In the packing LP, we get a binary solution with $c = e$, but not with $c = (1, 1, 1, 1, 1, 0, 0, 0, 0)$, with the “ones” corresponding to vertices $\{r, s, t, u, v\}$: set $x_r = x_s = x_t = x_u = x_v = \frac{1}{2}$. 