Integer and Combinatorial Optimization: Totally Unimodular Matrices

John E. Mitchell

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

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Definition

An $m \times n$ matrix $A$ is **totally unimodular (TU)** if the determinant of each square submatrix is equal to 0, 1, or -1.

See Nemhauser and Wolsey, section III.1.2, for more information.
Necessary conditions

**Theorem**

If \( A \) is totally unimodular then all the vertices of \( \{ x \in \mathbb{R}_+^n : Ax = b \} \) are integer for any integer vector \( b \in \mathbb{R}^m \).

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It follows that if \( A \) is totally unimodular then the optimal solution to the integer program

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\begin{align*}
\text{max} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0, \text{ integer}
\end{align*}
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can be found by solving its LP relaxation.
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Theorem

The following statements are equivalent:

1. A is totally unimodular.
2. $A^T$ is totally unimodular.
4. A matrix obtained by deleting a unit row or column of $A$ is totally unimodular.
5. A matrix obtained by multiplying a row or column of $A$ by $-1$ is totally unimodular.
6. A matrix obtained by interchanging two rows or two columns of $A$ is totally unimodular.
7. A matrix obtained by duplicating rows or columns of $A$ is totally unimodular.
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**Note:** unit rows and columns are rows and columns of the identity matrix.
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Sufficient conditions

**Theorem**

An integer matrix $A$ with every entry $a_{ij} = 0$ or $\pm 1$ is totally unimodular if no more than two nonzero entries appear in any column, and if the rows of $A$ can be partitioned into two sets $I_1$ and $I_2$ such that:

1. If a column has two nonzero entries of the same sign then their rows are in different sets.
2. If a column has two nonzero entries of opposite signs then their rows are in the same set.
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Problems on graphs

Corollary

Any linear program of the form

$$\begin{align*}
\text{max } & \quad c^T x \\
\text{subject to } & \quad Ax = b \quad \text{or} \quad \text{subject to } \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}$$

where $A$ is either

1. the node-arc incidence matrix of a directed graph, or
2. the node-edge incidence matrix of an undirected bipartite graph

has only integer optimal vertices.

Thus, the following problems can be solved by solving linear programs, and the optimal solutions to the LPs are integral:

shortest path, max-flow, assignment, weighted bipartite matching, ...
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Another sufficient condition

**Theorem**

If all the extreme points of \( \{ x \in \mathbb{R}^n_+ : Ax \leq b \} \) are integral for all \( b \in \mathbb{Z}^m \), then \( A \) is totally unimodular.

Note that it is not true that if all the vertices of the polyhedron

\[ \{ x \in \mathbb{R}^n_+ : Ax = b \} \]

are integral for all \( b \in \mathbb{Z}^m \), then \( A \) is totally unimodular. Consider for example

\[ A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \text{det} A = -2. \]

Solve \( Ax = b \):

\[ x_1 = b_1 - b_2 - 2b_3 \]
\[ x_2 = b_2 + b_3 \]
\[ x_3 = b_3 \]
Necessary and sufficient condition

Theorem

The following statements are equivalent:

1. A is totally unimodular.
2. For every $J \subseteq N := \{1, \ldots, n\}$, there exists a partition $J_1, J_2$ of $J$ such that

$$| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} | \leq 1 \quad \text{for } i = 1, \ldots, m.$$ 

Note that the first Theorem on sufficient conditions is a special case of this theorem (after transposing the matrix).
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An example

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A = \begin{bmatrix}
1 & 1 & -1 & 0 \\
1 & -1 & 0 & -1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

When \( J = \{1, 2, 4\} \) we can take \( J_1 = \{1, 4\} \) and \( J_2 = \{2\} \).
But there is no partition that works when \( J = \{1, 2\} \).

\[\left( \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right) \leq 1\]