Integer and Combinatorial Optimization: Finding a Violated Disjunctive Inequality

John E. Mitchell

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

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Disjunctions

Let

$$Q = \{ x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq e \}, \quad S = Q \cap \mathbb{Z}^n$$

and

$$S^0_j = \{ x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq e, x_j = 0 \} = \{ x \in Q : x_j = 0 \}$$
$$S^1_j = \{ x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq e, x_j = 1 \} = \{ x \in Q : x_j = 1 \}$$

for $j = 1, \ldots, n$.

Here, $e$ denotes the vector of all ones.
Solve an LP to find a cut

Note that integrality is not enforced in $S_j^0$ and $S_j^1$. We show how constraints can be generated one at a time by solving a linear program.

Let

$$Q_1 = \text{conv}(S_i^0 \cup S_i^1).$$
An example

We take

\[ S = \{ x \in \mathbb{Z}^2 : 2x_1 - 2x_2 \leq 1, \ 0 \leq x_1, x_2 \leq 1 \}. \]

We want to exploit the disjunction \((x_1 = 0) \lor (x_1 = 1)\) to find a constraint \(\pi^T x \leq \pi_0\) that is valid for \(Q_1\) but violated by the point \((0.5, 0)\).
What do we need from the cut?

The new constraint must be valid for both $S_0^1$ and $S_1$. The polyhedron $S_0^1$ consists of $x \in \mathbb{R}^2$ satisfying

\[
\begin{align*}
2x_1 - 2x_2 &\leq 1 & \lambda_1 \\
-x_1 &\leq 0 & \lambda_2 \\
-x_2 &\leq 0 & \lambda_3 \\
x_1 &\leq 0 & \lambda_4 \\
x_2 &\leq 1 & \lambda_5
\end{align*}
\]

Any valid inequality for $S_0^1$ is either a nonnegative combination of these inequalities, or it is dominated by a nonnegative combination of them.

With nonnegative multipliers $\lambda_i$ as indicated, a nonnegative combination has the form

\[
(2\lambda_1 - \lambda_2 + \lambda_4)x_1 + (-2\lambda_1 - \lambda_3 + \lambda_5)x_2 \leq \lambda_1 + \lambda_5
\]
Valid constraints for $S_1^0$

With nonnegative multipliers $\lambda_i$ as indicated, a nonnegative combination has the form

$$(2\lambda_1 - \lambda_2 + \lambda_4)x_1 + (-2\lambda_1 - \lambda_3 + \lambda_5)x_2 \leq \lambda_1 + \lambda_5$$

For the inequality

$$\pi_1 x_1 + \pi_2 x_2 \leq \pi_0$$

to be valid for $S_1^0$, we need $\pi$ and $\lambda$ to satisfy

$$\begin{align*}
\pi_1 &\leq 2\lambda_1 - \lambda_2 + \lambda_4 \\
\pi_2 &\leq -2\lambda_1 - \lambda_3 + \lambda_5 \\
\pi_0 &\geq \lambda_1 + \lambda_5
\end{align*}$$
Valid constraints for $S_1^1$

Similarly, the polyhedron $S_1^1$ consists of $x \in \mathbb{R}^2$ satisfying

$$
egin{align*}
2x_1 & - 2x_2 & \leq & 1 & \mu_1 \\
-x_1 & & \leq & -1 & \mu_2 \\
-x_2 & & \leq & 0 & \mu_3 \\
x_1 & & \leq & 1 & \mu_4 \\
x_2 & & \leq & 1 & \mu_5
\end{align*}
$$

Any valid inequality for $S_1^1$ is either a nonnegative combination of these inequalities, or it is dominated by a nonnegative combination of them.

With nonnegative multipliers $\mu_i$ as indicated, a nonnegative combination has the form

$$(2\mu_1 - \mu_2 + \mu_4)x_1 + (-2\mu_1 - \mu_3 + \mu_5)x_2 \leq \mu_1 - \mu_2 + \mu_4 + \mu_5$$
Requirements on the multipliers

With nonnegative multipliers $\mu_i$ as indicated, a nonnegative combination has the form

$$(2\mu_1 - \mu_2 + \mu_4)x_1 + (-2\mu_1 - \mu_3 + \mu_5)x_2 \leq \mu_1 - \mu_2 + \mu_4 + \mu_5$$

For the inequality

$$\pi_1 x_1 + \pi_2 x_2 \leq \pi_0$$

to be valid for $S_1^0$, we need $\pi$ and $\mu$ to satisfy

$$\begin{align*}
\pi_1 & \leq 2\mu_1 - \mu_2 + \mu_4 \\
\pi_2 & \leq -2\mu_1 - \mu_3 + \mu_5 \\
\pi_0 & \geq \mu_1 - \mu_2 + \mu_4 + \mu_5
\end{align*}$$
Subproblem objective

We have determined conditions on \((\pi, \pi_0)\) so that the constraint 
\[\pi^T x \leq \pi_0\] 
is valid for \(Q_1\).

This new constraint should be violated at \(\bar{x} = (0.5, 0)\), so we use the
objective function

\[\max_{\pi, \pi_0} \pi^T \bar{x} - \pi_0\]

This value is positive if and only if the new constraint is violated by \(\bar{x}\).
Normalization

All the constraints on \((\pi, \pi_0)\) discussed so far are homogeneous, so the set of feasible solutions is a cone.

To get a bounded LP, we introduce a normalization constraint, giving a slice through the cone.

One possible normalization is to require

\[
\sum_{j=1}^{5} \lambda_j + \sum_{j=1}^{5} \mu_j = 1.
\]

Note that \(\lambda \geq 0\) and \(\mu \geq 0\) and any nontrivial solution must have at least one positive component of \(\lambda\) and \(\mu\), so any feasible solution is a rescaling of one satisfying the normalization constraint.
Cut generation LP

Putting this together, we search for a constraint by solving the cut generation linear program (CGLP)

\[
\begin{align*}
\max_{\pi, \pi_0, \lambda, \mu} & \quad 0.5\pi_1 - \pi_0 \\
\text{subject to} & \quad \pi_1 \leq 2\lambda_1 - \lambda_2 + \lambda_4 \\
& \quad \pi_2 \leq -2\lambda_1 - \lambda_3 + \lambda_5 \\
& \quad \pi_0 \geq \lambda_1 + \lambda_5 \\
& \quad \pi_1 \leq 2\mu_1 - \mu_2 + \mu_4 \\
& \quad \pi_2 \leq -2\mu_1 - \mu_3 + \mu_5 \\
& \quad \pi_0 \geq \mu_1 - \mu_2 + \mu_4 + \mu_5 \\
\sum_{j=1}^{5} \lambda_j + \sum_{j=1}^{5} \mu_j & = 1 \\
\lambda, \mu & \geq 0
\end{align*}
\]

Using AMPL, we find the solution to this LP is

\[
\begin{align*}
\pi_1 & = \frac{1}{3}, \quad \pi_2 = -\frac{2}{3}, \quad \pi_0 = 0 \\
\mu_1 & = \mu_2 = \frac{1}{3}, \quad \lambda_4 = \frac{1}{3}, \quad \text{other components of } \lambda, \mu \text{ are zero}
\end{align*}
\]
The cut

Using AMPL, we find the solution to this LP is

\[ \pi_1 = \frac{1}{3}, \quad \pi_2 = -\frac{2}{3}, \quad \pi_0 = 0 \]
\[ \mu_1 = \mu_2 = \frac{1}{3}, \quad \lambda_4 = \frac{1}{3}, \text{ other components of } \lambda, \mu \text{ are zero} \]

The resulting constraint \( \pi^T x \leq \pi_0 \) is a rescaling of the facet \( x_1 - 2x_2 \leq 0 \) of \( Q_1 \).
The cut generation LP is large, so we try to limit the number of constraints and variables.

Additional improvements are possible by trying to generate multiple disjunctive inequalities from a single disjunction by slightly perturbing $\bar{x}$. 
Limit number of $\pi$ variables

We can choose to only include $\pi$ variables for basic variables in the fractional solution to the original problem.

In our example, this would give a cut generation LP:

$$\max_{\pi_1, \pi_0, \lambda, \mu} 0.5\pi_1 - \pi_0$$

subject to

$$\pi_1 \leq 2\lambda_1 - \lambda_2 + \lambda_4$$
$$\pi_0 \geq \lambda_1$$
$$\pi_1 \leq 2\mu_1 - \mu_2 + \mu_4$$
$$\pi_0 \geq \mu_1 - \mu_2 + \mu_4$$
$$\lambda_1 + \lambda_2 + \lambda_4 + \mu_1 + \mu_2 + \mu_4 = 1$$
$$\lambda, \mu \geq 0$$

Note that $\pi_2$ is not included since $x_2$ is nonbasic at $\bar{x}$.

This also results in the omission of $\lambda_3, \lambda_5, \mu_3, \mu_5$ since they are dual multipliers for constraints that only involve $x_2$. 
Optimal solution to smaller CGLP

The optimal solution here is

\[ \pi_1 = \frac{1}{3}, \; \pi_0 = 0 \]
\[ \mu_1 = \mu_2 = \frac{1}{3}, \; \lambda_4 = \frac{1}{3}, \; \text{other components of } \lambda, \; \mu \text{ are zero} \]

giving the constraint \( \frac{1}{3} x_1 \leq 0 \).

Note that \( \lambda \) and \( \mu \) are exactly as before; the old solution will always be feasible in the new cut generation LP, but in general it might not be optimal.

This constraint is only valid when \( x_2 = 0 \), that is, when \( x_2 \) remains nonbasic.
Lift to get valid cut when $x_2 > 0$

We need to lift to get a valid constraint for the whole problem, so solve a further subproblem:

$$\max \left\{ \frac{1}{3} x_1 : x \in Q, x_2 = 1 \right\}.$$ 

The optimal solution is $\hat{x} = (1, 1)$ with value $\frac{1}{3}$, so the lifted constraint is

$$\frac{1}{3} x_1 - \frac{1}{3} x_2 \leq 0.$$ 

Note that this constraint is actually stronger than the one we derived before, because we have exploited integrality of $x_2$ in the lifting process. In general, this approach recovers constraints as good as the original process.
Only include active constraints

We can choose to only include constraints that are active in the original problem.

In the example, the only inactive constraint is the upper bound constraint on $x_2$, which had multipliers $\lambda_5$ and $\mu_5$ in the CGLP.

Both of these multipliers were zero in the optimal solution to CGLP, so the generated cut would have been the same even if we omitted them.
A modified example

Omitting inactive constraints may affect the final solution. We take

\[ S = \{ x \in \mathbb{Z}^2 : 2x_1 - 2x_2 \leq 1, 19x_1 + 10x_2 \leq 10, 0 \leq x_1, x_2 \leq 1 \}. \]

Omitting the inactive constraint \( 19x_1 + 10x_2 \leq 10 \) leads to the same cut as before, \( x_1 - 2x_2 \leq 0 \). Including the inactive constraint gives the stronger disjunctive cut \( x_1 \leq 0 \).