Integer and Combinatorial Optimization: The Christofides Heuristic

John E. Mitchell

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

February 2019
The Traveling Salesman Problem

Given a graph \( G = (V, E) \), a \textbf{Hamiltonian tour} is a cycle that contains all of the nodes.

If each edge \( e \) has a length \( d_e \), the \textbf{traveling salesman problem} is to find the tour with least total length.
The Christofides heuristic is a polynomial time algorithm that constructs a tour.

If the edge lengths satisfy the triangle inequality then the heuristic is guaranteed to be within 50% of optimality.

This is the best known bound for any polynomial heuristic for the class of traveling salesman problems satisfying the triangle inequality.
The Christofides Heuristic

1. Given a complete graph with vertices $V$ and edge lengths $d_e$ for $e \in E$.
2. Find a minimum weight spanning tree $S$ on the graph.
3. Let $U \subseteq V$ be the vertices with odd degree in $S$.
4. Find a minimum weight perfect matching $M$ on the vertices $U$.
5. Find an Eulerian walk on the edges of $M \cup S$.
6. If any node is visited more than once, shortcut the tour.
7. The resulting collection of edges is a tour.
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An example

spanning tree
An example

nodes with odd degree
An example

matching
An example

Eulerian walk

Mitchell

The Christofides Heuristic
An example

Shortcut at node 3
An example

Hamiltonian path
Why does this give a tour?

Since the number of edges in the spanning tree $S$ is $|V| - 1$, the total degree of the nodes in the spanning tree is $2(|V| - 1)$, an even number. Therefore, there must be an even number of nodes with odd degree, so a perfect matching exists.

Every vertex has even degree in the connected set of edges $S \cup M$, so an Eulerian walk exists, that is, a walk that traverses each edge in $M \cup S$ exactly once.

Note that some edges may be in both $M$ and $S$, so these edges are traversed once as part of $M$ and once as part of $S$. 
The triangle inequality

In the example, the edge lengths are the Euclidean distances between the vertices. These all satisfy the triangle inequality:

\[ d_{rt} \leq d_{rs} + d_{st} \quad \text{for any vertices } r, s, t \in V \]

We assume all edge lengths satisfy the triangle inequality.

This is satisfied, for example, by graphs corresponding to destinations on a plane with distances equal to Euclidean distances.

Theorem

The Christofides heuristic produces a tour that is no more than 50% longer than the length of the optimal tour.
Proof of performance guarantee: the tree

Let $T^*$ be an optimal tour, with length $l(T^*)$.

The length $l(S)$ of the minimum spanning tree $S$ is no greater than the length of the optimal tour, since every tour contains a spanning tree.

Thus $l(S) \leq l(T^*)$. 
Proof of performance guarantee: the matching

The length $l(M)$ of the optimal matching $M$ is no greater than half the length of the optimal tour:

a tour can be broken into two perfect matchings (plus one additional edge if the number of vertices is odd).

Since the triangle inequality holds, the matching $M$ is at least as good as the better of these two matchings obtained from the tour.

Thus, $l(M) \leq \frac{1}{2} l(T^*)$. 

Break tour into 2 parts.
Shortcut to get 2 perfect matchings.
Proof of performance guarantee: overall

Since the triangle inequality holds, shortcutting the tour to avoid visiting a vertex more than once can only shorten the length of the tour. Thus, the tour $T$ found by the heuristic has length $l(T)$ satisfying

$$l(T) \leq l(S) + l(M).$$

Therefore, $l(T) \leq \frac{3}{2} l(T^*)$. □
Achieving the worst case bound

In the notation of the proof, to have $I(T) = \frac{3}{2} I(T^*)$ we would have to have $I(S) = I(T^*)$ and $I(M) = \frac{1}{2} I(T^*)$.

If all the edge lengths are positive, we will always have $I(S) < I(T^*)$, so we will not achieve the worst-case bound.

We can get arbitrarily close, as in the following example.
Approaching the worst case

Let $n$ be odd and arrange the vertices in two lines. The distance between neighbors in the same line is $1 + \epsilon$ for some small positive $\epsilon$; the distance between neighbors in different lines is 1.
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nodes with odd degree
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Diagram:

1 2 3 4 5 6

matching, and Hamiltonian tour
Length of tour

The length of this tour is

\[ l(S) = n - 1 \quad \text{for the edges in the spanning tree} \]
\[ l(M) = (1 + \epsilon) \frac{n-1}{2} \quad \text{for the edge in the matching} \]
\[ l(T) = (3 + \epsilon) \frac{n-1}{2} \quad \text{for the tour} \]
A worst-case example

Optimal tour

This has length

\[ l(T^*) = 2 + (n - 2)(1 + \epsilon) \]

The ratio of these tour lengths is

\[
\frac{l(T)}{l(T^*)} = \frac{(3 + \epsilon)(n - 1)}{2(2 + (n - 2)(1 + \epsilon))} \rightarrow \frac{3}{2}
\]

as \( n \rightarrow \infty \), \( \epsilon \rightarrow 0 \).