Integer and Combinatorial Optimization: Valid Inequalities for Knapsack Problems

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Binary knapsack problem

The binary knapsack problem is

$$\max_{x \in \mathbb{B}^n} \quad c^T x$$
subject to \quad $$a^T x \leq b$$

We assume \(a_j \leq b\) for \(j = 1, \ldots, n\)
(otherwise, if \(a_j > b\) then any feasible solution has \(x_j = 0\)).

Let \(S\) be the set of feasible solutions.

We have \(\dim(\text{conv}(S)) = n\), since the origin is feasible, as are all the unit vectors.
A cover $C \subseteq \{1, \ldots, n\}$ is a subset of the indices satisfying

$$\sum_{j \in C} a_j > b.$$ 

Any valid cover $C$ gives the valid inequality

$$\sum_{j \in C} x_j \leq |C| - 1.$$
Minimal cover inequalities

If $C \setminus k$ is not a cover for any $k \in C$ then $C$ is a minimal cover.

Theorem

Let $S^0 := \{x \in S : x_j = 0 \text{ if } j \notin C\}$. If $C$ is a minimal cover then the cover inequality defines a facet of $S^0$.

Proof.

(Sketch) Each of the leave-one-out binary vectors is on the face and these vectors are linearly independent.

If a cover is not minimal then it does not define a facet:

it is implied by a minimal cover inequality together with upper bound constraints $x_j \leq 1$. 

$8x_1 + 3x_1 + 3x_2 + 4x_4 \leq 9$

Eq: $C = \{2, 3, 4\}, \{1, 2, 3\}, \{1, 3\}, \{1, 4\}$
**Minimal cover inequalities**

\[ 8x_1 + 3x_2 + 3x_3 + 4x_4 \leq 9 \]

If \( C \setminus k \) is not a cover for any \( k \in C \) then \( C \) is a **minimal cover**.

**Theorem**

Let \( S^0 := \{ x \in S : x_j = 0 \text{ if } j \notin C \} \). If \( C \) is a minimal cover then the cover inequality defines a facet of \( S^0 \).

**Proof.**

(Sketch) Each of the leave-one-out binary vectors is on the face and these vectors are linearly independent.

If a cover is **not** minimal then it does not define a facet:

it is implied by a minimal cover inequality together with upper bound constraints \( x_j \leq 1 \).

*Eg:* \( C = \{1, 2, 3, 4\} \) \( \text{If } x_1 + x_2 + x_3 + x_4 = 3 \) \( x_1, x_2, x_3, x_4 \leq 1 \) **impossible**.
\[2x_1 + 3x_2 + 3x_3 + 4x_4 \leq 9\]
\[C = \{2, 3, 4\} \text{ minimal.}\]

Larger cover: \(\{1, 2, 3, 4\}\).

If \(x_1 + x_2 + x_3 + x_4 = 3\)
then have: (i) \(x_1 = 1\)
and (ii) \(x_1 + x_2 + x_4 = 2\)
Lifting cover inequalities

**Example 1:** The set $C = \{2, 3, 4\}$ is a minimal cover for the knapsack constraint

$$8x_1 + 3x_2 + 3x_3 + 4x_4 \leq 9,$$

giving the valid constraint

$$x_2 + x_3 + x_4 \leq 2.$$

To lift on $x_1$, we solve the subproblem

$$\max_{x \in \mathbb{B}^4} \quad x_2 + x_3 + x_4$$

subject to

$$8x_1 + 3x_2 + 3x_3 + 4x_4 \leq 9$$

$$x_1 = 1$$
Example 1 continued

To lift on $x_1$, we solve the subproblem

$$\max_{x \in \mathbb{B}^4} x_2 + x_3 + x_4$$
subject to

$$8x_1 + 3x_2 + 3x_3 + 4x_4 \leq 9$$
$$x_1 = 1$$

Taking $x_1 = 1$ forces $x_2 = x_3 = x_4 = 0$ so the optimal value of the lifting subproblem is 0, so the lifting coefficient for $x_1$ is $2 - 0$, so the lifted constraint is

$$2x_1 + x_2 + x_3 + x_4 \leq 2.$$ 

Note that we had to solve a knapsack problem to find the lifting coefficient. The LP relaxation of the lifting subproblem has optimal value $\frac{1}{3}$, so solving this instead would have led to the slightly weaker constraint

$$\frac{5}{3}x_1 + x_2 + x_3 + x_4 \leq 2.$$
Example 2

The set $C = \{1, 2, 3\}$ is a minimal cover for the knapsack constraint

$$3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 \leq 11,$$

giving the valid constraint

$$x_1 + x_2 + x_3 \leq 2.$$

We lift first on $x_4$, so we solve the subproblem

$$\max_{x \in \mathbb{B}^5} x_1 + x_2 + x_3$$

subject to

$$3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 \leq 11$$

$$x_4 = 1, x_5 = 0$$

This has optimal value 2, so the lifting coefficient for $x_4$ is $2 - 2 = 0$. The lifted constraint is

$$x_1 + x_2 + x_3 \leq 2.$$
Example 2 continued

We next lift on $x_5$, so we solve the subproblem

$$\max_{x \in \mathbb{B}^5} x_1 + x_2 + x_3$$

subject to

$$3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 \leq 11$$
$$x_5 = 1$$

This has optimal value 2, so the lifting coefficient for $x_5$ is also $2 - 2 = 0$.

The lifted constraint is

$$x_1 + x_2 + x_3 \leq 2.$$
Lifting cover inequalities

**Theorem**

If $C$ is a minimal cover then lifting the cover inequality gives a facet for $S$.

This follows from our earlier theorem about maximal liftings, since the dimension of the feasible region increases by 1 each time an extra variable is added.