Valid Inequalities for Knapsack Problems

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1 The Binary Knapsack Problem

The binary knapsack problem is

\[
\max_{x \in \{0,1\}^n} \quad c^T x \\
\text{subject to} \quad a^T x \leq b
\]

We assume \( a_j \leq b\) for \( j = 1, \ldots, n\) (otherwise, if \( a_j > b\) then any feasible solution has \( x_j = 0\)). Let \( S\) be the set of feasible solutions. We have \( \dim(\text{conv}(S)) = n\), since the origin is feasible, as are all the unit vectors.

2 Cover inequalities

A cover \( C \subseteq \{1, \ldots, n\} \) is a subset of the indices satisfying

\[
\sum_{j \in C} a_j > b.
\]

Any valid cover \( C \) gives the valid inequality

\[
\sum_{j \in C} x_j \leq |C| - 1.
\]

If \( C \setminus k \) is not a cover for any \( k \in C \) then \( C \) is a minimal cover.

**Theorem 1** Let \( S^0 := \{ x \in S : x_j = 0 \text{ if } j \notin C \}\). If \( C \) is a minimal cover then the cover inequality defines a facet of \( S^0\).

**Proof.** (Sketch) Each of the leave-one-out binary vectors is on the face and these vectors are linearly independent. \(\square\)

If a cover is not minimal then it does not define a facet: it is implied by a minimal cover inequality together with upper bound constraints \( x_j \leq 1\).

3 Lifted cover inequalities

We can lift minimal cover inequalities to give facets of \( S\).

**Example 1:** The set \( C = \{2, 3, 4\} \) is a minimal cover for the knapsack constraint

\[
8x_1 + 3x_2 + 3x_3 + 4x_4 \leq 9,
\]

giving the valid constraint

\[
x_2 + x_3 + x_4 \leq 2.
\]
To lift on $x_1$, we solve the subproblem

$$\max_{x \in \mathbb{B}^4} \quad x_2 + x_3 + x_4$$

subject to

$$8x_1 + 3x_2 + 3x_3 + 4x_4 \leq 9$$
$$x_1 = 1$$

Taking $x_1 = 1$ forces $x_2 = x_3 = x_4 = 0$ so the optimal value of the lifting subproblem is 0, so the lifting coefficient for $x_1$ is $2 - 0$, so the lifted constraint is

$$2x_1 + x_2 + x_3 + x_4 \leq 2.$$ 

Note that we had to solve a knapsack problem to find the lifting coefficient. The LP relaxation of the lifting subproblem has optimal value $\frac{1}{3}$, so solving this instead would have led to the slightly weaker constraint

$$\frac{5}{3}x_1 + x_2 + x_3 + x_4 \leq 2.$$ 

**Example 2:** The set $C = \{1, 2, 3\}$ is a minimal cover for the knapsack constraint

$$3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 \leq 11,$$

giving the valid constraint

$$x_1 + x_2 + x_3 \leq 2.$$ 

We lift first on $x_4$, so we solve the subproblem

$$\max_{x \in \mathbb{B}^5} \quad x_1 + x_2 + x_3$$

subject to

$$3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 \leq 11$$
$$x_4 = 1, \quad x_5 = 0$$

This has optimal value 2, so the lifting coefficient for $x_4$ is $2 - 2 = 0$. The lifted constraint is

$$x_1 + x_2 + x_3 \leq 2.$$ 

We next lift on $x_5$, so we solve the subproblem

$$\max_{x \in \mathbb{B}^5} \quad x_1 + x_2 + x_3$$

subject to

$$3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 \leq 11$$
$$x_5 = 1$$

This has optimal value 2, so the lifting coefficient for $x_5$ is also $2 - 2 = 0$. The lifted constraint is

$$x_1 + x_2 + x_3 \leq 2.$$ 

**Theorem 2** If $C$ is a minimal cover then lifting the cover inequality gives a facet for $S$.

This follows from our earlier theorem about maximal liftings, since the dimension of the feasible region increases by 1 each time an extra variable is added.