Integer and Combinatorial Optimization: Lifting Inequalities

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Lifting is a general procedure for strengthening a valid inequality. We let $S$ denote a set of binary points and consider two subsets that constitute a partition of $S$:

\[
S^0 = \{ x \in S : x_1 = 0 \}
\]
\[
S^1 = \{ x \in S : x_1 = 1 \}
\]
Inequalities on $S^0$ and $S$

We assume the inequality

$$\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$  \hspace{1cm} (1)$$

is valid for $S^0$. We’d like to extend this inequality so it is also valid for $S^1$, looking at constraints of the form

$$\alpha x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$ \hspace{1cm} (2)$$

It will be useful to examine

$$\zeta := \max \left\{ \sum_{j=2}^{n} \pi_j x_j : x \in S^1 \right\}. \hspace{1cm} (3)$$
Overestimating $\zeta$

If we assume we know a polyhedron $P$ such that $S = P \cap \mathbb{B}^n$ then we can find an overestimate for $\zeta$ by solving the LP relaxation:

$$\max \left\{ \sum_{j=2}^{n} \pi_j x_j : \ x \in P, \ x_1 = 1 \right\}.$$
The lifted inequality is valid

**Theorem**

If \( \alpha \leq \pi_0 - \zeta \) then (2) is valid for \( S \).

**Proof.**

We break into cases:

- \( x \in S^0 \): then \( x_1 = 0 \) so (2) holds since (1) holds on \( S^0 \).

- \( x \in S^1 \): then \( \sum_{j=2}^{n} \pi_j x_j \leq \zeta \), so

  \[
  \alpha x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \alpha + \zeta \leq \pi_0
  \]

  from the definition of \( \zeta \).

Hence the inequality is valid for all \( x \in S \).
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Hence the inequality is valid for all \( x \in S \).
The lifted inequality increases dimension

**Theorem**

Assume (1) defines a face of dimension $k$ of $S^0$. If $\alpha = \pi_0 - \zeta$ then (2) defines a face of dimension at least $k + 1$ of $S$.

**Proof.**

We give $k + 2$ affinely independent vectors that satisfy (2) at equality:

- take $x_1 = 0$ along with each of $k + 1$ affinely independent vectors in $S^0$ that satisfy (1) at equality.
- take $x_1 = 1$ along with a point that solves the problem

$$\max \{ \sum_{j=2}^{n} \pi_j x_j : x \in S^1 \}.$$ 

These vectors are affinely independent (exercise).
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The lifted inequality can be facet defining

**Corollary**

*If (1) defines a facet of $S^0$ and if $\dim(S) = \dim(S^0) + 1$ then (2) defines a facet of $S$.***

Note that this gives us another method for showing a valid inequality defines a facet.
Sequential lifting

Let \( N = \{1, \ldots, n\} \). The lifting procedure should be used \textit{sequentially}:

if the inequality \( \sum_{j \in N_1} \pi_j x_j \leq \pi_0 \) is valid for 
\( S \cap \{ x \in \mathbb{B}^n : x_j = 0, j \in N \setminus N_1 \} \),

we can lift on variables \( x_j \in N \setminus N_1 \) one at a time to get a valid inequality

\[
\sum_{j \in N \setminus N_1} \alpha_j x_j + \sum_{j \in N_1} \pi_j x_j \leq \pi_0
\]

valid for all of \( S \).
Changing the order of lifting

If the variables are lifted in a different order, a different inequality may be obtained. For example, the odd hole constraint

\[ x_2 + \ldots + x_6 \leq 2 \]

can be lifted first on \( x_1 \) and then on \( x_7 \), or vice versa, for the following graph: