Lifting Inequalities

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Lifting is a general procedure for strengthening a valid inequality. We let $S$ denote a set of binary points and consider two subsets that constitute a partition of $S$:

$$
S^0 = \{ x \in S : x_1 = 0 \} \\
S^1 = \{ x \in S : x_1 = 1 \}
$$

We assume the inequality

$$
\sum_{j=2}^{n} \pi_j x_j \leq \pi_0
$$

is valid for $S^0$. We’d like to extend this inequality so it is also valid for $S^1$, looking at constraints of the form

$$
\alpha x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0
$$

It will be useful to examine

$$
\zeta := \max \left\{ \sum_{j=2}^{n} \pi_j x_j : x \in S^1 \right\}.
$$

If we assume we know a polyhedron $P$ such that $S = P \cap \mathbb{B}^n$ then we can find an overestimate for $\zeta$ by solving the LP relaxation:

$$
\max \left\{ \sum_{j=2}^{n} \pi_j x_j : x \in P, x_1 = 1 \right\}.
$$

**Theorem 1** If $\alpha \leq \pi_0 - \zeta$ then (2) is valid for $S$.

**Proof.** We break into cases:

1. $x \in S^0$: then $x_1 = 0$ so (2) holds since (1) holds on $S^0$.

2. $x \in S^1$: then $\sum_{j=2}^{n} \pi_j x_j \leq \zeta$, so

$$
\alpha x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \alpha + \zeta \leq \pi_0
$$

from the definition of $\zeta$.

Hence the inequality is valid for all $x \in S$. $\square$
Theorem 2 Assume (1) defines a face of dimension $k$ of $S^0$. If $\alpha = \pi_0 - \zeta$ then (2) defines a face of dimension at least $k+1$ of $S$.

Proof. We give $k + 2$ affinely independent vectors that satisfy (2) at equality:

- take $x_1 = 0$ along with each of $k + 1$ affinely independent vectors in $S^0$ that satisfy (1) at equality.
- take $x_1 = 1$ along with a point that solves the problem

$$\max \{ \sum_{j=2}^{n} \pi_j x_j : x \in S^1 \}.$$  

These vectors are affinely independent (exercise).

Corollary 1 If (2) defines a facet of $S^0$ and if $\dim(S) = \dim(S^0) + 1$ then (2) defines a facet of $S$.

Note that this gives us another method for showing a valid inequality defines a facet.

Let $N = \{1, \ldots, n\}$. The lifting procedure should be used sequentially.

If the inequality $\sum_{j \in N^1} \pi_j x_j \leq \pi_0$ is valid for $S \cap \{x \in \mathbb{B}^n : x_j = 0, j \in N \setminus N^1\}$, we can lift on variables $x_j \in N \setminus N^1$ one at a time to get a valid inequality

$$\sum_{j \in N \setminus N_1} \alpha_j x_j + \sum_{j \in N^1} \pi_j x_j \leq \pi_0$$

valid for all of $S$.

If the variables are lifted in a different order, a different inequality may be obtained. For example, the odd hole constraint

$$x_2 + \ldots + x_6 \leq 2$$

can be lifted first on $x_1$ and then on $x_7$, or vice versa, for the following graph: