We show how the dimension of a face can be determined using the construction of affinely independent points. This approach can be used to prove that a face of a polyhedron is a facet. Later, we will demonstrate another method for the MaxCut problem.

Recall two equivalent definitions of affine independence:

**Definition 1** The \( k+1 \) points \( a^0, a^1, \ldots, a^k \in \mathbb{R}^n \) are affinely independent if the \( k \) vectors \( a^1 - a^0, a^2 - a^0, \ldots, a^k - a^0 \) are linearly independent.

**Definition 2** The \( k+1 \) points \( a^0, a^1, \ldots, a^k \in \mathbb{R}^n \) are affinely independent if the only solution \( \lambda_0, \lambda_1, \ldots, \lambda_k \) to the system

\[
\sum_{i=0}^{k} \lambda_i a^i = 0, \quad \sum_{i=0}^{k} \lambda_i = 0
\]

is \( \lambda_0 = \lambda_1 = \ldots = \lambda_k = 0 \).

Note that Definition 2 implies that if \( k+1 \) points are linearly independent then they are also affinely independent. (The converse does not necessarily hold.)

We also have the following proposition:

**Proposition 1** If a set \( S \subseteq \mathbb{R}^n \) contains \( k+1 \) affinely independent points then the dimension of \( S \) is at least \( k \).

Thus, constructing a sufficiently large set of affinely independent points in \( S \) provides a lower bound on the dimension of \( S \). If we can construct valid equalities \( Ax = b \) that are satisfied by all points in \( S \) then we obtain an upper bound of \( n - \text{rank}(A) \) on the dimension of \( S \). Getting these bounds to agree would then give the dimension of \( S \).

When \( S \) is a set of integer (or binary) points, the first step is to determine the dimension \( d \) of \( S \). To then show that an inequality \( g^T x \geq h \) defines a facet of \( S \), we need to:

- Show every point \( x \) in \( S \) satisfies \( g^T x \geq h \), so the inequality is valid.
- Find \( d \) affinely independent points in \( S \) satisfying \( g^T x = h \), so the dimension of the face is at least \( d - 1 \).
- Find one point \( x \in S \) satisfying \( g^T x > h \), so the inequality defines a proper face of \( S \).

This approach is used for node packing in a graph on \( n \) vertices.

First, the convex hull of the set \( S \) of feasible packings can be shown to have dimension \( n \): we have \( n+1 \) affinely independent points in \( S \), namely the origin and all the unit vectors.

Then there exist \( n \) affinely independent points in \( S \) satisfying \( x_i = 0 \) for any \( i = 1, \ldots, n \). Further, there exist \( n \) affinely independent points in \( S \) satisfying \( \sum_{i \in C} x_i = 1 \) for any maximal clique \( C \) in the graph.