Integer and Combinatorial Optimization: Strong Valid Inequalities for Structured Integer Programs

John E. Mitchell

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

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The node packing problem

Given a graph $G = (V, E)$, a node packing is a subset $U \subseteq V$ of the vertices such that no pair in the set is joined by an edge. We model this by introducing a binary variable

$$x_i = \begin{cases} 1 & \text{if node } i \text{ is in the packing} \\ 0 & \text{otherwise} \end{cases}$$

The set of feasible solutions is

$$S = \{ x \in \mathbb{B}^n : x_i + x_j \leq 1 \text{ for all } (i, j) \in E \},$$

where $n = |V|$, so $S$ is the set of incidence vectors of node packings. We want to find facets of $\text{conv}(S)$, the convex hull of $S$. A facet is a face of dimension one less than the dimension of $\text{conv}(S)$. 
An example

![Graph Diagram](image-url)
An example

\[ U = \{2, 4\} \]
An example

\[ U = \{3\} \]
An example

$U = \{1, 5, 7\}$
Proving the dimension of a set

Recall:

**Definition**
The \( k + 1 \) vectors \( x^0, x^1, \ldots, x^k \in \mathbb{R}^n \) are *affinely independent* if the \( k \) vectors \( x^1 - x^0, x^2 - x^0, \ldots, x^k - x^0 \) are linearly independent.

**Definition**
A polyhedron \( P \subseteq \mathbb{R}^n \) has dimension at least \( k \) if it contains \( k + 1 \) affinely independent vectors.

We want to determine the dimension of \( \text{conv}(S) \). We know \( \text{conv}(S) \subseteq \mathbb{R}^n \), so the dimension \( \dim(\text{conv}(S)) \leq n \).

We can show \( \dim(\text{conv}(S)) = n \) if we can find \( n + 1 \) affinely independent vectors in \( \text{conv}(S) \). Since \( S \subseteq \text{conv}(S) \), it suffices to find \( n + 1 \) affinely independent vectors in \( S \).
The dimension of conv(S)

Proving the dimension of a set

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The $k + 1$ vectors $x^0, x^1, \ldots, x^k \in \mathbb{R}^n$ are **affinely independent** if the $k$ vectors $x^1 - x^0, x^2 - x^0, \ldots, x^k - x^0$ are linearly independent.

**Definition**

A polyhedron $P \subseteq \mathbb{R}^n$ has dimension at least $k$ if it contains $k + 1$ affinely independent vectors.

We want to determine the dimension of conv($S$). We know conv($S$) $\subseteq \mathbb{R}^n$, so the dimension dim(conv($S$)) $\leq n$.

We can show dim(conv($S$)) = $n$ if we can find $n + 1$ affinely independent vectors in conv($S$). Since $S \subseteq$ conv($S$), it suffices to find $n + 1$ affinely independent vectors in $S$. 

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Strong Valid Inequalities
The dimension of \( \text{conv}(S) \)

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**Definition**

A polyhedron $P \subseteq \mathbb{R}^n$ has dimension at least $k$ if it contains $k + 1$ affinely independent vectors.

We want to determine the dimension of $\text{conv}(S)$. We know $\text{conv}(S) \subseteq \mathbb{R}^n$, so the dimension $\dim(\text{conv}(S)) \leq n$.

We can show $\dim(\text{conv}(S)) = n$ if we can find $n + 1$ affinely independent vectors in $\text{conv}(S)$. Since $S \subseteq \text{conv}(S)$, it suffices to find $n + 1$ affinely independent vectors in $S$. 
Affinely independent packings

We have the following set of vectors in $S$:

1. $S = \emptyset$: taking no vertices is a node packing. This gives $x^0 = 0$, the zero vector.

2. $S = \{i\}$ for $i = 1, \ldots, n$. Taking any single vertex is a node packing (assuming the graph is simple). This gives $x^i = e^i$, the $i$th unit vector for $i = 1, \ldots, n$.

The vectors $x^0, x^1, \ldots, x^n$ are affinely independent, since the vectors $x^1 - x^0, \ldots, x^n - x^0$ are the $n$ unit vectors, which are linearly independent. Hence we have the following result.

**Proposition**

The set $S$ of incidence vectors of node packings on a graph $G = (V, E)$ with $n$ vertices is full-dimensional, with dimension equal to $n$. 

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Nonegative inequalities

For each vertex \(i\), the nonnegativity constraint \(x_i \geq 0\) is valid for \(S\), since it is satisfied by every incidence vector.

What is the dimension of the corresponding face?

For a fixed \(i\), the face is

\[
F^i = \{ x \in \text{conv}(S) : x_i = 0 \},
\]

the points in \(\text{conv}(S)\) that satisfy the constraint at equality. We know the dimension of the face is no greater than \(n - 1\), since it sits within the hyperplane \(x_i = 0\).

To show it has dimension equal to \(n - 1\), we need to find \(n\) affinely independent points on the face; that is, \(n - 1\) node packings all satisfying \(x_i = 0\).
A non-maximal clique

Packings with \( x_5 = 0 \)?
A non-maximal clique

Packings with $x_5 = 0$: $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{6\}, \{7\}$
$n$ packings on the face $x_i = 0$

Such a set is the following:

1. $S = \emptyset$: taking no vertices is a node packing. This gives $x^0 = 0$, the zero vector.

2. $S = \{j\}$ for $j = 1, \ldots, n$, $j \neq i$. Taking any single vertex other than $i$ is a node packing that does not contain $i$ (assuming the graph is simple). This gives $x^j = e^j$, the $j$th unit vector for $j = 1, \ldots, n, j \neq i$. These $n$ vectors are affinely independent, so the face has dimension equal to $n - 1 = \dim(\text{conv}(S)) - 1$.

Proposition

The nonnegative inequalities define facets of $\text{conv}(S)$. 

Nonnegative inequalities

$n$ packings on the face $x_i = 0$

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Clique inequalities

Any edge \((i, j) \in E\) implies the valid inequality \(x_i + x_j \leq 1\). This can be generalized to give a valid inequality for any clique \(C \subseteq V\):

\[
\sum_{k \in C} x_k \leq 1
\]

This is valid because any two vertices in \(C\) are adjacent.

Any single vertex is a clique of size 1.

An edge corresponds to a clique of size 2.

A clique \(C\) is maximal if \(C \cup \{v\}\) is not a clique for any \(v \in V \setminus C\).
The clique $C = \{4, 5, 6\}$ is not maximal. Any node packing including either vertex 4 or vertex 5 or vertex 6 cannot include vertex 3. Thus, any packing satisfying $x_4 + x_5 + x_6 = 1$ must also have $x_3 = 0$. 
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A non-maximal clique

The clique $C = \{4, 5, 6\}$ is not maximal. Any node packing including either vertex 4 or vertex 5 or vertex 6 cannot include vertex 3. Thus, any packing satisfying $x_4 + x_5 + x_6 = 1$ must also have $x_3 = 0$. 
**Lemma**

*If a clique $C$ is not maximal then the clique inequality does not define a facet of $\text{conv}(S)$.***

**Proof.**

Since the clique is not maximal it is part of a larger clique, so $\exists v \in V \setminus C$ where $C \cup \{v\}$ is a clique. Then any packing that satisfies $\sum_{k \in C} x_k = 1$ selects a vertex in $C$, so it cannot select $v$ which is adjacent to all these vertices, so it must also satisfy $x_v = 0$.

These two equality constraints are linearly independent, so the dimension of the face of $\text{conv}(S)$ given by the clique inequality is no larger than $n - 2$. 


Clique inequalities

Cliques that are not maximal

Lemma

If a clique $C$ is not maximal then the clique inequality does not define a facet of $\text{conv}(S)$.

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Since the clique is not maximal it is part of a larger clique, so $\exists v \in V \setminus C$ where $C \cup \{v\}$ is a clique. Then any packing that satisfies $\sum_{k \in C} x_k = 1$ selects a vertex in $C$, so it cannot select $v$ which is adjacent to all these vertices, so it must also satisfy $x_v = 0$.

These two equality constraints are linearly independent, so the dimension of the face of $\text{conv}(S)$ given by the clique inequality is no larger than $n - 2$. 

\[\square\]
For example, the upper bound constraint $x_i \leq 1$ does not define a facet if $i$ is incident to at least one edge.

The edge constraint $x_i + x_j \leq 1$ does not define a facet if $i$ and $j$ are both members of the same larger clique.
Maximal cliques
We will show that the clique constraints corresponding to maximal cliques define facets of conv(S). The result requires the following lemma, which we will justify in the next handout.

Lemma
If the vectors $x^1, \ldots, x^k$ are linearly independent then they are affinely independent.

We can now state and prove the main result.

Proposition
If $C$ is a maximal clique then the clique inequality

$$\sum_{k \in C} x_k \leq 1$$

defines a facet of conv(S).
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We will show that the clique constraints corresponding to maximal cliques define facets of conv($S$). The result requires the following lemma, which we will justify in the next handout.

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Packings satisfying the constraint at equality

Clique $C = \{1, 2, 3\}$ is a maximal clique.

The incidence vectors of these seven packings are linearly independent.
Clique inequalities

Packings satisfying the constraint at equality

Clique $C = \{1, 2, 3\}$ is a maximal clique. Node packings that include exactly one vertex from $C$ include the following seven:

$$\{1\},$$

The incidence vectors of these seven packings are linearly independent.
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Packings satisfying the constraint at equality

Want
\[ x_1 + x_2 + x_3 = 1 \]

Clique \( C = \{1, 2, 3\} \) is a maximal clique. Node packings that include exactly one vertex from \( C \) include the following seven:

\[ \{1\}, \{2\}, \]

The incidence vectors of these seven packings are linearly independent.
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\{1\}, \{2\}, \{3\}, \{4, 2\}, \{5, 1\}, \{6, 2\}, \{7, 1\}
\]

The incidence vectors of these seven packings are linearly independent.
Proof that maximal cliques give facets (part 1)

We need to find $n$ affinely independent incident vectors of packings that satisfy the constraint at equality.

We construct such packings in two ways:

1. For each $i \in C$, choose the packing $\{i\}$, giving the incidence vector $e_i$.

2. For each $j \not\in C$, there is a vertex $i \in C$ that is not adjacent to $j$ since $C$ is maximal. Choose the packing $\{i, j\}$, giving the incidence vector $x_i + x_j$. 
Proof that maximal cliques give facets (part 1)

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We construct such packings in two ways:

1. For each \( i \in C \), choose the packing \( \{i\} \), giving the incidence vector \( e^i \).

2. For each \( j \notin C \), there is a vertex \( i \in C \) that is not adjacent to \( j \) since \( C \) is maximal. Choose the packing \( \{i, j\} \), giving the incidence vector \( x^i + x^j \).
Continue the proof

These $n$ incidence vectors are linearly independent. One way to see this is to write them out explicitly. By reordering if necessary, we assume $C$ consists of vertices $1, \ldots, p$, where $|C| = p$.

\[
\begin{bmatrix}
  x^1 & \ldots & x^p & x^{p+1} + x^{i_1} & \ldots & x^n + x^{i_{n-p}} \\
  1 & 0 & 0 & \uparrow & \ldots & \uparrow \\
  0 & 0 & 0 & e^{i_1} & \downarrow & e^{i_{n-p}} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 1 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & \ddots & 1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]
The proof (part 3)

These $n$ incidence vectors are the columns of the $n \times n$ matrix

$$Q = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix}$$

for some matrix $M$.

The columns of $Q$ are linearly independent.

Hence the vectors are affinely independent, so the constraint defines a facet of $\text{conv}(S)$. 

\[\square\]