Integer and Combinatorial Optimization: 
Cuts for Mixed Integer Problems

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Our integer program

We consider a constraint involving just two variables, one integral and one continuous. In particular, we have the constraint

\[ y \leq b + x \]

where \( y \) is a nonnegative integral variable, \( x \) is a nonnegative continuous variable, and \( b \) is a scalar parameter.

The parameter \( b \) is nonintegral, and we set

\[ f = b - \lfloor b \rfloor > 0. \]
The feasible region

We can visualize the feasible region:
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\[ y = b + x \]

\[ (0, b) \]

\[ (1 - f, \lfloor b \rfloor) \]

\[ (0, \lfloor b \rfloor) \]
The point \((0, b)\) satisfies the constraint \(y \leq b + x\), but it is not in the convex hull of feasible solutions.

When \(y = \lfloor b \rfloor\), we must have \(x \geq 0\).

When \(y = \lceil b \rceil\), we must have \(x \geq \lceil b \rceil - b = 1 - f\).

The line that goes through the two points \((0, \lfloor b \rfloor)\) and \((1 - f, \lceil b \rceil)\) gives the valid constraint

\[ y \leq \lfloor b \rfloor + \frac{1}{1 - f}x. \]
Another version

We can use a similar argument to generate a cutting plane from the constraint

\[ y + x \geq b \]

where again \( y \) is a nonnegative integral variable, \( x \) is a nonnegative continuous variable, and \( b \) is a scalar parameter.

The parameter \( b \) is nonintegral, and we set

\[ f = b - \lfloor b \rfloor > 0. \]
The feasible region

We can visualize the feasible region:

The valid inequality is

\[
y + \frac{1}{f}x \geq \lceil b \rceil
\]
The feasible region

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The valid inequality is

\[ y + \frac{1}{f} x \geq \lceil b \rceil \]
Fixed charge networks

In a fixed charge network, we have arcs that incur a cost if the flow on the arc is positive.

The charge is the same regardless of the size of the flow.

The arcs also have capacities.

Let

\[ x_j = \text{flow on arc } j \text{ (nonnegative, continuous variable)} \]

\[ y_j = \begin{cases} 
1 & \text{if use arc } j \\
0 & \text{otherwise}
\end{cases} \]

\[ a_j = \text{capacity of arc } j. \]
Relate $y_j$ and $x_j$

We can force $y_j$ to take the correct value by imposing the constraint

$$x_j \leq a_j y_j.$$
A set of arcs

Now consider the situation where we have a set of arcs $C$ entering a vertex, which has maximum possible demand $b$.

We immediately have the flow conservation constraint

$$\sum_{j \in C} x_j \leq b.$$ 

We assume

$$\lambda := \sum_{j \in C} a_j - b > 0$$

We can find additional constraints that involve the fixed charge variables $y$. 
Flow cover constraint

![Diagram of a network flow problem](image)

We have

\[
\sum_{j \in C} x_j \leq b, \quad \lambda := \sum_{j \in C} a_j - b > 0
\]

We can find additional constraints that involve the fixed charge variables \( y \).

In particular, we have the constraint

\[
\sum_{j \in C} x_j \leq b - \sum_{j \in C} (a_j - \lambda)^+ (1 - y_j)
\]

where the superscript \( + \) indicates the positive part of its argument.
Validity of the flow cover constraint

\[ \sum_{j \in C} x_j \leq b - \sum_{j \in C} (a_j - \lambda)^+ (1 - y_j), \quad \lambda := \sum_{j \in C} a_j - b > 0 \]

Why is this valid? We consider the various cases.

- All \( y_j = 1 \): then the constraint reduces to the flow conservation constraint.
- Exactly one \( y_j = 0 \) with \( a_j \geq \lambda \): Then the right hand side becomes equal to the sum of the capacities of the arcs with \( y_j = 1 \).
- More than one \( y_j = 0 \): The capacity of the open arcs is

\[ \sum_{j \in C: y_j = 1} a_j = b + \lambda - \sum_{j \in C: y_j = 0} a_j \leq b - \sum_{j \in C: y_j = 0} (a_j - \lambda)^+ \]

as required.
Validity of the flow cover constraint

\[ \sum_{j \in C} x_j \leq b - \sum_{j \in C} (a_j - \lambda)^+ (1 - y_j), \quad \lambda := \sum_{j \in C} a_j - b > 0 \]

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\[ \sum_{j \in C : y_j = 1} a_j = b + \lambda - \sum_{j \in C : y_j = 0} a_j \leq b - \sum_{j \in C : y_j = 0} (a_j - \lambda)^+ \]

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\[ \sum_{j \in C} x_j \leq b - \sum_{j \in C} (a_j - \lambda)^+(1 - y_j), \quad \lambda := \sum_{j \in C} a_j - b > 0 \]

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- More than one \( y_j = 0 \): The capacity of the open arcs is
  \[ \sum_{j \in C: y_j = 1} a_j = b + \lambda - \sum_{j \in C: y_j = 0} a_j \leq b - \sum_{j \in C: y_j = 0} (a_j - \lambda)^+ \]
  as required.
An example

For example, if we have three arcs with capacities \( a_1 = 5, a_2 = 6, a_3 = 7 \) and we have \( b = 15 \) then we get \( \lambda = 18 - 15 = 3 \).

The flow cover constraint is

\[
x_1 + x_2 + x_3 \leq 15 - 2(1 - y_1) - 3(1 - y_2) - 4(1 - y_3)
\]

The constraint is designed to prevent fractional solutions that are not in the convex hull of feasible solutions.

For example, we could take \( x_1 = 5, x_2 = 3, x_3 = 7 \), which meets the demand.

The simple upper bound constraint \( x_j \leq a_j y_j \) would only force \( y_1 = 1, y_2 = \frac{1}{2}, y_3 = 1 \).

However, this solution violates the flow cover constraint: the left hand side evaluates to 15, while the right hand side evaluates to \( 13\frac{1}{2} \).
Further extensions

This constraint can be found in Proposition 4.2 on page 282 of Nemhauser and Wolsey.

It can be extended to more complicated situations; see, for example, Proposition 4.3 on page 283 and equation (4.5) on page 499.