1 Mixed integer rounding

We consider a constraint involving just two variables, one integral and one continuous. In particular, we have the constraint

\[ y \leq b + x \]

where \( y \) is a nonnegative integral variable, \( x \) is a nonnegative continuous variable, and \( b \) is a scalar parameter. The parameter \( b \) is nonintegral, and we set

\[ f = b - \lfloor b \rfloor > 0. \]

We can visualize the feasible region:

The point \((0, b)\) satisfies the constraint \( y \leq b + x \), but it is not in the convex hull of feasible solutions. When \( y = \lfloor b \rfloor \), we must have \( x \geq 0 \). When \( y = \lceil b \rceil \), we must have \( x \geq \lceil b \rceil - b = 1 - f \). The line that goes through the two points \((0, \lfloor b \rfloor)\) and \((1 - f, \lceil b \rceil)\) gives the valid constraint

\[ y \leq \lfloor b \rfloor + \frac{1}{1-f} x. \]
We can use a similar argument to generate a cutting plane from the constraint
\[ y + x \geq b \]
where again \( y \) is a nonnegative integral variable, \( x \) is a nonnegative continuous variable, and \( b \) is a scalar parameter. The parameter \( b \) is nonintegral, and we set
\[ f = b - \lfloor b \rfloor > 0. \]

We can visualize the feasible region:

The valid inequality is
\[ y + \frac{1}{f}x \geq \lfloor b \rfloor \]
2 Flow cover inequalities

In a fixed charge network, we have arcs that incur a cost if the flow on the arc is positive. The charge is the same regardless of the size of the flow. The arcs also have capacities. Let

\[ x_j = \text{flow on arc } j \text{ (nonnegative, continuous variable)} \]
\[ y_j = \begin{cases} 1 & \text{if use arc } j \\ 0 & \text{otherwise} \end{cases} \]
\[ a_j = \text{capacity of arc } j. \]

We can force \( y_j \) to take the correct value by imposing the constraint

\[ x_j \leq a_j y_j. \]

Now consider the situation where we have a set of arcs \( C \) entering a vertex, which has maximum possible demand \( b \).

We immediately have the flow conservation constraint

\[ \sum_{j \in C} x_j \leq b. \]

We assume

\[ \lambda := \sum_{j \in C} a_j - b > 0 \]

We can find additional constraints that involve the fixed charge variables \( y \). In particular, we have the constraint

\[ \sum_{j \in C} x_j \leq b - \sum_{j \in C} (a_j - \lambda)^+(1 - y_j) \]

where the superscript + indicates the positive part of its argument.

Why is this valid? We consider the various cases.

- All \( y_j = 1 \): then the constraint reduces to the flow conservation constraint.
- Exactly one \( y_j = 0 \) with \( a_j \geq \lambda \): Then the right hand side becomes equal to the sum of the capacities of the arcs with \( y_j = 1 \).
- More than one \( y_j = 0 \): The capacity of the open arcs is

\[ \sum_{j \in C: y_j = 1} a_j = b + \lambda - \sum_{j \in C: y_j = 0} a_j \leq b - \sum_{j \in C: y_j = 0} (a_j - \lambda)^+ \]

as required.

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For example, if we have three arcs with capacities $a_1 = 5$, $a_2 = 6$, $a_3 = 7$ and we have $b = 15$ then we get $\lambda = 18 - 15 = 3$. The flow cover constraint is

$$x_1 + x_2 + x_3 \leq 15 - 2(1 - y_1) - 3(1 - y_2) - 4(1 - y_3)$$

The constraint is designed to prevent fractional solutions that are not in the convex hull of feasible solutions. For example, we could take $x_1 = 5$, $x_2 = 3$, $x_3 = 7$, which meets the demand. The simple upper bound constraint $x_j \leq a_j y_j$ would only force $y_1 = 1$, $y_2 = \frac{1}{2}$, $y_3 = 1$. However, this solution violates the flow cover constraint: the left hand side evaluates to 15, while the right hand side evaluates to $13\frac{1}{2}$.

This constraint can be found in Proposition 4.2 on page 282 of Nemhauser and Wolsey. It can be extended to more complicated situations; see, for example, Proposition 4.3 on page 283 and equation (4.5) on page 499.