Integer and Combinatorial Optimization: Gomory's Cutting Plane Algorithm

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Our integer program

We take our standard problem to be the integer program

$$\begin{array}{lrcl} \min_{x \in \mathbb{R}^n} & c^T x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \\ & x & & \text{integer} \end{array} \qquad (IP)$$

We propose to solve this problem by repeating the following loop:

- Solve the LP relaxation.
- If integral, STOP.
- Otherwise, find a violated Gomory cutting plane, add it to the LP relaxation, and loop.

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A variable for the objective function value

Gomory cutting planes can be generated from any basic variable that takes a non-integral value in the solution to the current LP relaxation.

They can also be generated from the objective function, assuming all entries in c are integral so the objective function value is integral.

To make it easier to talk about the objective function, we introduce a new integral variable x_0 equal to the objective function value, and solve the problem

$$\begin{array}{lrcl} \min_{(x_0,x)\in\mathbb{R}^{n+1}} & x_0 & & \\ \text{subject to} & x_0 - c^T x & = & 0 \\ & Ax & = & b \\ & x & \geq & 0 \\ & x & & \text{integer} \\ & x_0 & & \text{free} \end{array}$$

The LP relaxation

The LP relaxation is

$$\begin{array}{llcll} \min_{(x_0,x)\in\mathbb{R}^{n+1}} & x_0 & & & \\ \text{subject to} & x_0 - c^T x & = & 0 & \\ & Ax & = & b & \qquad (LP) \\ & x & \geq & 0 & \\ & x_0 & & \text{free} & \end{array}$$

Since x_0 is a free variable, it is always basic.

Gomory cutting plane

We let $\mathcal{B}$ denote the set of basic variables, so $\mathcal{B} \subseteq \{0, 1, \ldots, n\}$.

Let $\mathcal{R} = \{0, 1, \ldots, n\} \setminus \mathcal{B}$ denote the set of nonbasic variables.

If the optimal solution to (LP) is non-integral then there is a basic variable x_i that takes a fractional value.

We can write the row of the tableau corresponding to x_i as

$$x_i + \sum_{j \in \mathcal{R}} \bar{a}_{ij} = \bar{b}_i$$

We define

$$f_0 := \bar{b}_i - \lfloor \bar{b}_i \rfloor > 0, \quad f_j := \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \geq 0 \; \forall j \in \mathcal{R}$$

The Gomory cutting plane is then

$$\sum_{j \in \mathcal{R}} f_j x_j \geq f_0.$$

Reoptimize

The Gomory cutting plane is then

$$\sum_{j \in \mathcal{R}} f_j x_j \geq f_0.$$

For example, if x_2 is the basic variable in the following constraint

$$x_2 + 1.2x_4 - 0.7x_7 = 2.5$$

then we get the Gomory cut

$$0.2x_4 + 0.3x_7 \geq 0.5.$$

We introduce a (nonnegative integral) slack variable into the Gomory cut, add it to the LP relaxation, and reoptimize.

Convergence

Gomory proved that this algorithm converges to the optimal solution in a finite number of iterations, provided we obey some rules when breaking ties.

The most important rule is:

> *always add a cutting plane from the* **least index** *fractional basic variable.*

The pivot rules are called lexicographic rules.

Theorem (Gomory [2])

Using lexicographic pivot rules and always deriving the cut from the least index non-integral basic variable, Gomory's cutting algorithm converges to an optimal solution to (IP) in a finite number of iterations.

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Practical performance

For many years, Gomory cutting planes were regarded as inefficient. That changed in the 1990's, starting with the work of Balas et al. [1].

One simple refinement was that they added multiple cuts at once, corresponding to many of the integer variables taking non-integral values in the optimal solution to the current relaxation.

Gomory cuts are now one of the principal cuts included in CPLEX and other commercial packages.

Zanette et al. [3] revisited Gomory's lexicographic rules and showed that they are useful in practice: without the rules, the generated cuts are often almost parallel, which results in slow progress.

References

E. Balas, S. Ceria, G. Cornuéjols, and N. Natraj.
Gomory cuts revisited.
Operations Research Letters, 19:1–9, 1996.

R. E. Gomory.
An algorithm for integer solutions to linear programs.
In R. L. Graves and P. Wolfe, editors, *Recent Advances in Mathematical Programming*, pages 269–302. McGraw-Hill, New York, 1963.

A. Zanette, M. Fischetti, and E. Balas.
Lexicography and degeneracy: can a pure cutting plane algorithm work?
Mathematical Programming, 130(1):153–176, 2011.

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