Integer and Combinatorial Optimization:
Gomory Cuts for Matching Problems

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February 2019
Minimum weight perfect matching

Let $G = (V, E)$ be a graph with edge weights $w_e$ for $e \in E$. Consider the problem of finding a minimum weight perfect matching on the following graph:

![Graph with edge weights](image_url)
IP formulation

The minimum weight perfect matching problem for this graph is

$$\min_x \sum_{e \in E} w_e x_e$$
subject to

- $x_{ab} + x_{ac} + x_{ad} = 1$
- $x_{ab} + x_{bc} + x_{bg} = 1$
- $x_{ac} + x_{bc} + x_{cf} = 1$
- $x_{df} + x_{dg} + x_{ad} = 1$
- $x_{df} + x_{fg} + x_{cf} = 1$
- $x_{dg} + x_{fg} + x_{bg} = 1$

$x_e$ binary $\forall e \in E$
The LP relaxation can be written as the following tableau:

\[
\begin{array}{cccccccc}
  x_{ab} & x_{ac} & x_{bc} & x_{df} & x_{dg} & x_{fg} & x_{ad} & x_{cf} & x_{bg} \\
  0 & 2 & 3 & 1 & 1 & 2 & 3 & 7 & 8 & 9 \\
  1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
  1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
  1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{array}
\]
Solution to LP relaxation

The optimal tableau is

<table>
<thead>
<tr>
<th></th>
<th>$x_{ab}$</th>
<th>$x_{ac}$</th>
<th>$x_{bc}$</th>
<th>$x_{df}$</th>
<th>$x_{dg}$</th>
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<th>$x_{ad}$</th>
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<tbody>
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<td>$-1/2$</td>
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</table>

Each of the constraints give the same Gomory cut, namely

$$\frac{1}{2}x_{ad} + \frac{1}{2}x_{cf} + \frac{1}{2}x_{bg} \geq \frac{1}{2}$$
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\[ \frac{1}{2}x_{ad} + \frac{1}{2}x_{cf} + \frac{1}{2}x_{bg} \geq \frac{1}{2} \]
Gomory cut
Each of the constraints give the same Gomory cut, namely
\[
\frac{1}{2} x_{ad} + \frac{1}{2} x_{cf} + \frac{1}{2} x_{bg} \geq \frac{1}{2}
\]
Denote the slack variable in this constraint by \( s \). Adding the constraint to the tableau and reoptimizing gives an optimal simplex tableau of

<table>
<thead>
<tr>
<th>( x_{ab} )</th>
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Since solution to LP relaxation is integral, it solves original problem. Minimum perfect matching: edges \((b, c)\), \((f, g)\), and \((a, d)\), value 11.
The optimal solution
Subtour elimination constraints

The Gomory cutting plane is a rescaling of the subtour elimination constraint

\[ x_{ad} + x_{cf} + x_{bg} \geq 1, \]

which is equivalent to

\[ x_{ab} + x_{ac} + x_{bc} \leq 1, \]

as can be determined by adding together the first three original constraints, each with weight \( \frac{1}{2} \).

\[
\begin{align*}
  x_{ab} + x_{ac} + x_{ad} &= 1 \\
  x_{ab} + x_{bc} + x_{bg} &= 1 \\
  x_{ac} + x_{bc} + x_{cf} &= 1 \\
  x_{ab} + x_{ac} + x_{bc} + \frac{1}{2} x_{ad} + \frac{1}{2} x_{bg} + \frac{1}{2} x_{cf} &= \frac{3}{2}
\end{align*}
\]

It is also equivalent to

\[ x_{df} + x_{dg} + x_{fg} \leq 1 : \]

add together the last three original constraints, each with weight \( \frac{1}{2} \).
Can't use more than 
\( \frac{|U|-1 \text{ edges in } E(U)}{2} \)

Add degree constraints for u \in U, with weight \( t \):

\[
\sum_{e \in E(U)} x_e + \frac{1}{2} \sum_{e \in S(U)} x_e = \frac{1}{2} |U|\]

in 2 degree constraints

\( \sum_{e \in E(U)} x_e \leq \frac{|U|-1}{2} \)

in 1 degree constraint in our sum