Gomory Cutting Planes

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RPI

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An integer program

We want to solve the integer program

$$\begin{align*}
\text{min} & \quad z := -x_1 - x_2 \\
\text{subject to} & \quad 2x_1 + 5x_2 \leq 20 \\
& \quad 4x_1 + 3x_2 \leq 17 \\
& \quad x_1, x_2 \geq 0, \text{ integer.}
\end{align*}$$

Notice that $z$ is also integral.
LP relaxation

We can introduce slack variables $x_3$ and $x_4$, which are both nonnegative integers. The LP relaxation is then in standard form with tableau

\[
\begin{array}{cccccc}
 & x_1 & x_2 & x_3 & x_4 \\
0 & -1 & -1 & 0 & 0 \\
20 & 2 & 5 & 1 & 0 \\
17 & 4 & 3 & 0 & 1 \\
\end{array}
\]
Solve the LP relaxation

Simplex pivots to make first $x_1$ and then $x_2$ basic give an optimal solution to the LP relaxation:

$$
\begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  \frac{17}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} \\
  \frac{23}{2} & 0 & \frac{7}{2} & 1 & -\frac{1}{2} \\
  \frac{17}{4} & 1 & \frac{3}{4} & 0 & \frac{1}{4} \\
\end{array}
$$

Thus the solution to the LP relaxation is $x^{*}_{LP} = \left( \frac{25}{14}, \frac{23}{7} \right)$. 
Graphing the LP relaxation

**Graph:**
- Axes: $x_1$ (horizontal) and $x_2$ (vertical).
- Line segment: $x_1 + x_2 = 4$.
- Point marked: $X_{LP}^*$.

**Notes:**
- $X_{LP}^*$ represents the optimal solution to the LP relaxation.

**Equations:**
- $x_1 + x_2 = 4$.

**References:**
- Mitchell (RPI).
- Gomory Cuts.
The constraints in the final simplex tableau are

\[
\begin{align*}
    x_2 + \frac{2}{7}x_3 - \frac{1}{7}x_4 &= \frac{23}{7} \\
    x_1 - \frac{3}{14}x_3 + \frac{5}{14}x_4 &= \frac{25}{14}
\end{align*}
\]

which can be written equivalently as

\[
\begin{align*}
    (x_2 - x_4) + \left(\frac{2}{7}x_3 + \frac{6}{7}x_4\right) &= 3 + \frac{2}{7} \\
    (x_1 - x_3) + \left(\frac{11}{14}x_3 + \frac{5}{14}x_4\right) &= 1 + \frac{11}{14}
\end{align*}
\]
Since all the variables are integer, we must have \((x_2 - x_4)\) and \((x_1 - x_3)\) integral, so the fractional parts on the right hand side must come from the fractional parts on the left hand side. As written, the fractional parts on the left hand sides are nonnegative, so we get the following \textbf{Gomory cutting planes} which are valid for the integer program:

\[
\begin{align*}
\frac{2}{7}x_3 + \frac{6}{7}x_4 & \geq \frac{2}{7} \\
\frac{11}{14}x_3 + \frac{5}{14}x_4 & \geq \frac{11}{14}
\end{align*}
\]

These coefficients are the \textbf{fractional parts} of the coefficients in the constraints in the final simplex tableau. Note that we only get \textit{inequalities}, since it is possible, for example, for \(x_2 - x_4 = 2\).
The inequalities can be written equivalently as

\[
x_2 - x_4 \leq 3 \\
x_1 - x_3 \leq 1
\]

where now the coefficients are the \textbf{round-downs} of the coefficients in the final simplex tableau.

\[
x_2 + \frac{2}{7} x_3 - \frac{1}{7} x_4 = \frac{23}{7}
\]

\underline{Round down:}

\[
x_2 + 0x_3 - x_4 \leq 3 \quad \left\{ \begin{array}{l}
\frac{2}{7} x_3 + \frac{6}{7} x_4 \geq \frac{2}{7}.
\end{array} \right.
\]
Gomory cut from the objective function

The **objective function** can be written

\[-z + \frac{1}{14} x_3 + \frac{3}{14} x_4 = \frac{71}{14}.\]

Exploiting integrality of \(z\) gives the Gomory cut

\[\frac{1}{14} x_3 + \frac{3}{14} x_4 \geq \frac{1}{14}.\]  

*Rounding version:*

\[-z \leq 5\]
Update the LP relaxation

All of the Gomory cuts are violated by the current basic feasible solution, which has $x_3 = x_4 = 0$. Any (or all) of the cuts can be added to the LP relaxation and the problem reoptimized. We choose to add (2), with slack variable $x_5$. Note that $x_5$ must also be integral. The updated tableau is:

$$
\begin{array}{ccccc}
X_1 & X_2 & X_3 & X_4 & X_5 \\
\hline
71/14 & 0 & 0 & 1/14 & 3/14 & 0 \\
23/7 & 0 & 1 & 2/7 & -1/7 & 0 \\
25/14 & 1 & 0 & -3/14 & 5/14 & 0 \\
-11/14 & 0 & 0 & -11/14 & -5/14 & 1 \\
\end{array}
$$

$$\frac{11}{14} X_3 + \frac{5}{14} X_4 \geq \frac{11}{14} \quad \left| \quad -\frac{11}{14} X_3 - \frac{5}{14} X_4 + X_5 = -\frac{11}{14} \right.$$
Solve the LP relaxation and the integer program

We reoptimize using dual simplex. We pivot on the last constraint, so $x_5$ leaves the basis and (from the minimum ratio test) $x_3$ enters the basis:

Thus, we get an optimal solution to the integer program: $x_{IP}^* = (2, 3)$, with value $z = -5$. 

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
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<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{2}{11}$</td>
<td>$\frac{1}{11}$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-\frac{3}{11}$</td>
<td>$\frac{4}{11}$</td>
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<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>$\frac{5}{11}$</td>
<td>$-\frac{3}{11}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\frac{5}{11}$</td>
<td>$-\frac{14}{11}$</td>
</tr>
</tbody>
</table>
Expressing the cut in the original variables

The Gomory cutting plane can be expressed in terms of the original variables $x_1$ and $x_2$. From the interpretation of $x_3$ and $x_4$ as slack variables in the original constraints, the cutting plane (2) is equivalent to

\[
\frac{11}{14} \leq \frac{11}{14} x_3 + \frac{5}{14} x_4
\]

\[= \frac{11}{14} (20 - 2x_1 - 5x_2) + \frac{5}{14} (17 - 4x_1 - 3x_2) \]

\[= \frac{305}{14} - \frac{21}{7} x_1 - \frac{70}{14} x_2 \]

\[= 21 \frac{11}{14} - 3x_1 - 5x_2 \]

or equivalently

\[3x_1 + 5x_2 \leq 21. \tag{4}\]
Expressing the other cuts in the original variables

Cutting planes (1) and (3) are scalings of one another. They can also be rewritten equivalently in terms of $x_1$ and $x_2$:

\[
1 \leq x_3 + 3x_4 \\
= (20 - 2x_1 - 5x_2) + 3(17 - 4x_1 - 3x_2) \\
= 71 - 14x_1 - 14x_2
\]

or equivalently

\[
x_1 + x_2 \leq 5. \quad (5)
\]
Graphing the Gomory cut

Gomory cut (5)

$\mathbf{x}_{IP}^*$

Gomory cut (4)
Mixed integer rounding

Strengthened versions of the Gomory cutting plane can be derived using logical arguments.

In particular, the **Gomory mixed integer rounding cut** can often be considerably stronger than the regular Gomory cut.