Definition 1. A node packing on a graph $G = (V, E)$ is a subset $U \subseteq V$ of the vertices so that no two vertices in $U$ are adjacent.

$U = \{4, 5\}$ is a node packing of cardinality 2

Definition 2. Let $G = (V, E)$ be a graph and let $z$ be a positive integer. The unweighted node packing with lower bound problem is to determine whether there is a node packing of cardinality at least $z$.

Theorem 1. The unweighted node packing with lower bound problem is $NP$-complete.

First show the problem is in $NP$:
Our certificate of feasibility consists of a list of the nodes in $U$. We can check quickly that none of these vertices are adjacent, and there are at least $z$ of them.

We show that 3-SAT can be polynomially transformed to our node packing problem.
We have an instance of 3-SAT with $n$ boolean variables $y_1, \ldots, y_n$ and $m$ clauses $C_1, \ldots, C_m$, with each clause containing exactly 3 literals. We construct a graph with $3m$ vertices which has a node packing of size at least $m$ if and only if the instance of 3-SAT is feasible.

A clique of size 3 is constructed corresponding to each clause. One vertex is constructed for each literal in the clause, and three edges are constructed so the three vertices are adjacent to each other.
Any packing of cardinality \( m \) must contain exactly one vertex from each of these \( m \) cliques. This vertex is a “representative” for the clause. We need to add additional edges to ensure consistency. Thus, if the representative for clause \( i \) is \( j_i \) then that corresponds to \( y_j = \text{TRUE} \), so there cannot be another chosen representative for another clause which corresponds to \( y_j = \text{FALSE} \).

Consistency can be ensured by adding edges of the form \((j_i, (n + j)_q)\) for any two clauses \( C_i \) and \( C_q \), where \( y_j \) is a literal in clause \( C_i \) and \( \bar{y}_j \) is a literal in clause \( C_q \).

**Example**

\[
\begin{align*}
C_1 &= y_1 + y_2 + \bar{y}_3 \\
C_2 &= y_2 + y_3 + \bar{y}_4 \\
C_3 &= \bar{y}_1 + \bar{y}_2 + y_4
\end{align*}
\]

There are many packings of cardinality 3, including \( U = \{1_1, 8_2, 6_3\} \). This corresponds to the valid truth assignment

\[
y_1 = \text{TRUE}, \ y_4 = \text{FALSE}, \ y_2 = \text{FALSE}, \ y_3 \text{ unspecified.}
\]

**Vertex cover**

Given a graph \( G = (V, E) \), a **vertex cover** is a subset \( U \subseteq V \) of the vertices so that every edge in \( E \) has at least one endpoint in \( U \). The vertex cover with upper bound problem is specified by a graph \( G = (V, E) \) and a scalar \( K \), and it is desired to find a vertex cover with cardinality \(|U| \leq K\).

**Exercise 1.** Show that vertex cover with upper bound is NP-complete, by transforming from node packing with lower bound.

**Hint:** If \( W \) is a node packing in graph \( G = (V, E) \) then its complement \( U = V \setminus W \) is a vertex cover.