1 The minimum spanning tree problem (MST)

We have a graph \( G = (V, E) \) with weight \( w_e \) for each edge \( e \in E \). We want to find a spanning tree \( T \subseteq E \) that has minimum weight \( \sum_{e \in T} w_e \).

Let \( n = |V| \), \( m = |E| \).

2 The number of feasible solutions

Lemma 1. For a complete graph on \( n \) vertices, the number of spanning trees is \( n^{n-2} \).

Proof. (Sketch)

Label the vertices \( 1, \ldots, n \). Let \( T \) be a spanning tree, so \( T \) has \( n-1 \) edges and at least two leaves. We construct a sequence of length \( n-2 \) using the following loop.

Initialize with \( S = \emptyset \). For \( i = 1, \ldots, n-2 \):

1. Let \( v_i \) be the lowest index leaf in the tree.
2. Let \( a_i \) denote the neighbor of \( v_i \) in the tree.
3. Add \( a_i \) to the end of the sequence \( S \).
4. Delete leaf \( v_i \) from \( T \).

The ordered string \( S \) consists of \( n-2 \) integers, each with a value between 1 and \( n \). Note that the last edge remaining in the tree after \( n-2 \) deletions is an edge between vertex \( n \) and one other vertex (vertex \( a_{n-2} \) if \( n \neq a_{n-2} \)).

It can also be shown that any sequence of \( n-2 \) integers valued between 1 and \( n \) (with repeats allowed) corresponds to a unique spanning tree.

Thus, we have a 1-to-1 correspondence between spanning trees and this set of sequences, which has cardinality \( n^{n-2} \).

Example with \( n = 6 \):

Sequence: \( 2 - 1 - 1 - 1 \).
3 Solving the MST

Minimum spanning tree problems can be solved using a **greedy algorithm**:

1. Initialize: $T = \emptyset$
2. Let $e$ be the edge with smallest weight that has not yet been considered.
3. If $T \cup \{e\}$ is acyclic, update $T \leftarrow T \cup \{e\}$; else delete $e$.
4. Return to Step 2.

Very quick sketch of proof: Let $\hat{T}$ be the tree returned by the greedy algorithm. Let $T^*$ be the optimal tree. Assume the shortest $k < n - 1$ edges in each tree are the same. Then we can add edge $e_{k+1}$ from $\hat{T}$ to $T^*$ and remove something on the fundamental cycle created in $T^* \cup e_{k+1}$. This tree will be at least as good as $T^*$ and will have the same shortest $k + 1$ edges as $\hat{T}$. By induction, we obtain that $\hat{T}$ is optimal. □

An example of the greedy algorithm

One solution: