When the primal problem has a unique nondegenerate optimal solution then so does the dual. In this handout we consider infeasible, degenerate and non-unique cases.

We work with the standard form linear program and its dual:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax = b, \quad x \geq 0 \quad (P) \\
\end{align*}
\]

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \quad (D) \\
\end{align*}
\]

where \( c \in \mathbb{R}^n, b \in \mathbb{R}^m, \) and \( A \in \mathbb{R}^{m \times n} \). Given \( y \in \mathbb{R}^m \), we define \( s := c - A^T y \in \mathbb{R}^n \). In what follows, we assume that the rows of \( A \) are linearly independent, so it has full row rank, so \( \text{rank}(A) = m \).

1 Dual rays and primal rays

Recall the Farkas Lemma:

Exactly one of the following systems has a solution:

(I) \( \exists x \in \mathbb{R}^n \) with \( Ax = b, x \geq 0 \),

(II) \( \exists d \in \mathbb{R}^m \) with \( A^T d \leq 0, b^T d > 0 \).

If \( (P) \) is infeasible then Farkas tells us that there must exist \( d \in \mathbb{R}^m \) with \( A^T d \leq 0 \) and \( b^T d > 0 \). Such a \( d \) is a dual ray and it gives a certificate of infeasibility for \( (P) \): for any feasible point \( \bar{y} \) for \( (D) \), the point \( \bar{y} + td \) is also feasible for any \( t > 0 \), and \( b^T (\bar{y} + td) \to \infty \) as \( t \to \infty \).

The set \( \{d \in \mathbb{R}^m : A^T d \leq 0\} \) is the recession cone for the feasible region for the dual problem (provided \( (D) \) is feasible). It is the set of directions we can move in from any feasible point for any nonnegative steplength and remain dual feasible.

We can also construct certificates of infeasibility for the dual problem \( (D) \). We have another Theorem of the Alternative:

Exactly one of the following systems has a solution:

(I) \( \exists d \) with \( Ad = 0, d \geq 0, c^T d < 0 \),

(II) \( \exists y \) with \( A^T y \leq c \).

Exercise: prove this using LP duality.

From this result, if the dual is infeasible then there exists a primal ray \( d \) with \( c^T d < 0 \). If \( \bar{x} \) is feasible in \( (P) \) then \( \bar{x} + td \) is feasible for any \( t \geq 0 \), and \( c^T (\bar{x} + td) \to -\infty \) as \( t \to \infty \).
2 Degeneracy and multiple optimal solutions

Let $B$ be an invertible $m \times m$ matrix whose columns are columns of $A$. Without loss of
generality, we can reorder the columns of $A$ so that the columns of $B$ are written first. We
denote the remaining columns by $N$, so we can then write the problem $(P)$ as

$$\begin{aligned}
\min & \quad c_B^T x_B + c_N^T x_N \\
\text{subject to} & \quad B x_B + N x_N = b \\
& \quad x_B, x_N \geq 0.
\end{aligned}$$

The dual can be written also in terms of $B$ and $N$:

$$\begin{aligned}
\max_y & \quad b^T y \\
\text{subject to} & \quad B^T y \leq c_B \\
& \quad N^T y \leq c_N
\end{aligned}$$

Take

$$\bar{x}_B = B^{-1} b, \quad \bar{x}_N = 0, \quad \bar{y} = B^{-T} c_B.$$ (1)

We assume $\bar{x}_B \geq 0$ and $N^T \bar{y} \leq c_N$, so this solution is primal and dual optimal.

2.1 Primal degeneracy

If $x_{B_i} = 0$ for some basic variable then we have a degenerate primal optimal solution. We
may well then have multiple optimal dual solutions, but this is not necessarily the case.

Lemma 1. Assume $\bar{x}$ has strictly fewer than $m$ positive components. If $N^T \bar{y} < c_N$ then the
dual problem has multiple optimal solutions.

Example 1. A problem with a degenerate optimal primal solution and a unique optimal dual
solution:

$$\begin{aligned}
\min_{x \in \mathbb{R}^4} & \quad x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 = 2 \\
& \quad x_1 + x_3 = 2 \\
& \quad x_2 + x_4 = 2 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

with dual

$$\begin{aligned}
\max_{y \in \mathbb{R}^3} & \quad 2y_1 + 2y_2 + 2y_3 \\
\text{subject to} & \quad y_1 + y_2 \leq 1 \\
& \quad y_1 + y_3 \leq 1 \\
& \quad y_2 \leq 0 \\
& \quad y_3 \leq 0
\end{aligned}$$

Every primal feasible point is optimal, including the degenerate solutions $x = (2, 0, 0, 2)$ and
$x = (0, 2, 2, 0)$. Also primal optimal is $\bar{x} = (1, 1, 1, 1)$, so by complementary slackness we
require $s = (0, 0, 0, 0)$ and hence the unique dual optimal solution is $\bar{y} = (1, 0, 0)$. 

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Lemma 2. If \((P)\) has a nondegenerate optimal solution then \((D)\) has a unique optimal solution.

Corollary 1. If \((D)\) has multiple optimal solutions then every optimal basic solution to \((P)\) is degenerate.

2.2 Dual degeneracy

A dual feasible solution \(\bar{y} = B^{-T}c_B\) is degenerate if more than \(m\) dual constraints hold exactly; equivalently, if fewer than \(n - m\) dual constraints hold strictly. Note we require \(y\) to have the form in (1) to define dual degeneracy. A nondegenerate dual solution \(\bar{y} = B^{-T}c_B\) is one where \(N^T\bar{y} < c_N\). We have analogous relationships between primal multiple optimal solutions and dual degeneracy to those in §2.1.

Lemma 3. If \((D)\) has a nondegenerate optimal solution then \((P)\) has a unique optimal solution.

Corollary 2. If \((P)\) has multiple optimal solutions then every optimal basic solution to \((D)\) is degenerate.

Lemma 4. Assume \(\bar{y}\) is a dual degenerate optimal solution. If \(\bar{x}_B > 0\) then the primal problem has multiple optimal solutions.

2.3 Primal and dual degeneracy

It is possible for both problems to have multiple degenerate optimal solutions as in the following example.

\[
\begin{align*}
\text{min}_x \quad & x_1 + x_2 + x_3 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 - x_4 = 4 \\
& 2x_1 + x_2 + 3x_3 - x_5 = 4 \\
& x_1 + x_2 + x_3 - x_6 = 2 \\
& x_1 \geq 0, \quad i = 1, \ldots, 6
\end{align*}
\]

Dual problem:
\[ \text{min}_{y,s} \quad 4y_1 + 4y_2 + 2y_3 \]
subject to
\[ 2y_1 + 2y_2 + y_3 + s_1 = 1 \]
\[ 3y_1 + y_2 + y_3 + s_2 = 1 \]
\[ y_1 + 3y_2 + y_3 + s_3 = 1 \]
\[ -y_1 + s_4 = 0 \]
\[ -y_2 + s_5 = 0 \]
\[ -y_3 + s_6 = 0 \]
\[ s_1 \geq 0, \quad i = 1, \ldots, 6 \]

3 Unbounded sets of optimal solutions

Either \((P)\) or \((D)\) (or both) can have an unbounded set of optimal solutions. In general, if one problem has an unbounded set of optimal solutions then it has multiple optimal solutions. It then follows from the earlier lemmas that every basic optimal solution to the other problem is degenerate. Consider the example problem

\[ \text{min}_{x \in \mathbb{R}^4} \quad x_1 \]
subject to
\[ 3x_1 + 2x_2 - x_3 + x_4 = 12 \]
\[ x_1, x_2, x_3, x_4 \geq 0 \]

Exercise: The following primal-dual pair has unbounded primal and dual optimal solutions:

\[ \text{min}_{x \in \mathbb{R}^3} \quad x_1 - x_2 \]
subject to
\[ x_1 - x_2 + x_3 = 0 \]
\[ x_3 = 0 \]
\[ x_1, x_2, x_3 \geq 0 \]

\[ \text{max}_{y \in \mathbb{R}^2} \quad 0 \]
subject to
\[ y_1 \leq 1 \]
\[ -y_1 \leq -1 \]
\[ y_1 + y_2 \leq 0 \]

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