1 The Diet Problem

We have daily requirements for $m$ nutrients: protein, carbohydrates, vitamin A, etc. To make things simple, we’ll assume all these requirements are lower bounds: we have to consume at least a certain amount $b_i$ of nutrient $i$ for $i = 1, \ldots, m$.

We have $n$ different foodstuffs we are prepared to eat. To simplify, we assume the foodstuffs are infinitely divisible, so we can use continuous variables. One unit of foodstuff $j$ costs $c_j$ and provides $a_{ij}$ units of nutrient $i$.

Minimizing our daily costs subject to meeting our nutritional requirements can then be expressed as a linear program:

$$\begin{align*}
\min_x & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, \ldots, m \\
& \quad x_j \geq 0 \quad j = 1, \ldots, n
\end{align*}$$

or equivalently

$$\begin{align*}
\min_x & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \\
& \quad x \geq 0.
\end{align*}$$

Now consider a manufacturer of nutrient powders, for $i = 1, \ldots, m$. How should a manufacturer choose prices $y_i$ for the powders so the consumer buys the powders instead of the foodstuffs? (Assume the consumer only cares about cost and is indifferent to taste.) To replace foodstuff $j$, need

$$\sum_{i=1}^{m} a_{ij} y_i \leq c_j.$$ 

The total revenue from sale of powders is $\sum_{i=1}^{m} b_i y_i$. Thus, the producer solves the following problem to select prices:

$$\begin{align*}
\max_y & \quad \sum_{i=1}^{m} b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad i = 1, \ldots, n \\
& \quad y_i \geq 0 \quad i = 1, \ldots, m
\end{align*}$$

or equivalently

$$\begin{align*}
\max_y & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \\
& \quad y \geq 0.
\end{align*}$$
2 Combinations of constraints

We want to get a lower bound on the optimal value of the linear program

\[
\begin{align*}
\min_x & \quad 7x_1 + 5x_2 + 8x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + 5x_3 \geq 6 \\
& \quad 3x_1 + 2x_2 + 4x_3 \geq 13 \\
& \quad 2x_1 + x_2 + 2x_3 \geq 8 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\] (1)

Let \( \bar{x} \) be feasible. From the first constraint, we must have, since \( \bar{x} \geq 0 \):

\[
7\bar{x}_1 + 5\bar{x}_2 + 8\bar{x}_3 \geq 1.5\bar{x}_1 + 3\bar{x}_2 + 7.5\bar{x}_3 = 1.5(\bar{x}_1 + 2\bar{x}_2 + 5\bar{x}_3) \geq 9.
\]

Similarly, from the second constraint, we must have:

\[
7\bar{x}_1 + 5\bar{x}_2 + 8\bar{x}_3 \geq 6\bar{x}_1 + 4\bar{x}_2 + 8\bar{x}_3 = 2(3\bar{x}_1 + 2\bar{x}_2 + 4\bar{x}_3) \geq 26.
\]

We can combine the constraints. Take the second and third constraints:

\[
7\bar{x}_1 + 5\bar{x}_2 + 8\bar{x}_3 \geq 7\bar{x}_1 + 4\bar{x}_2 + 8\bar{x}_3 = (3\bar{x}_1 + 2\bar{x}_2 + 4\bar{x}_3) + 2(2\bar{x}_1 + \bar{x}_2 + 2\bar{x}_3) \geq 13 + 16 = 29.
\]

Provided the scale factors are nonnegative, we get a valid lower bound. In particular, with weights \( y \) on the constraints, we need \( A^Ty \leq c \) to be able to conclude that \( b^Ty \) is a valid lower bound. The **dual problem** is to maximize this lower bound:

\[
\begin{align*}
\max_y & \quad b^Ty \\
\text{subject to} & \quad A^Ty \leq c \\
& \quad y \geq 0.
\end{align*}
\]

We can write this out explicitly as:

\[
\begin{align*}
\max_y & \quad 6y_1 + 13y_2 + 8y_3 \\
\text{subject to} & \quad y_1 + 3y_2 + 2y_3 \leq 7 \\
& \quad 2y_1 + 2y_2 + y_3 \leq 5 \\
& \quad 5y_1 + 4y_2 + 2y_3 \leq 8 \\
& \quad y_1, y_2, y_3 \geq 0.
\end{align*}
\] (2)

Note that \( \bar{x} = (3, 0, 1) \) is feasible in (1) with value 29, and \( \bar{y} = (0, 1, 2) \) is feasible in (2), also with value 29. Thus, \( \bar{x} \) and \( \bar{y} \) must solve their respective problems. This is **strong duality**: the optimal values agree (provided they exist).

Note also that at optimality, the **active constraints** in (1) correspond to the **positive components** in (2), and vice versa. This is an illustration of **complementary slackness**.