Simplex Iteration Information

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Our standard form

We work with the standard form linear program

$$\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax = b \quad (P) \\
& \quad x \geq 0
\end{align*}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$. In what follows, we assume that the rows of $A$ are linearly independent, so it has full row rank, so $\text{rank}(A) = m$. 
Use a basis matrix

Let $B$ be an invertible $m \times m$ matrix whose columns are columns of $A$. Without loss of generality, we can reorder the columns of $A$ so that the columns of $B$ are written first. We denote the remaining columns by $N$, so we write

$$A = \begin{bmatrix} B & N \end{bmatrix},$$

with $m$ columns and $n - m$ columns.

We similarly order the entries in $c$ and $x$, so

$$c = \begin{bmatrix} c_B \\ c_N \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_B \\ x_N \end{bmatrix},$$

with $c_B, x_B \in \mathbb{R}^m$ and $c_N, x_N \in \mathbb{R}^{n-m}$. 
Basic feasible solution

After a few manipulations, we can then write the problem \((P)\) as

\[
\begin{align*}
\min \quad & c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N \\
\text{subject to} \quad & x_B + B^{-1} N x_N = B^{-1} b \\
& x_B, \quad x_N \geq 0.
\end{align*}
\]

Note that

\[x_B = B^{-1} b - B^{-1} N x_N, \quad (1)\]

whatever the value of \(x_N\). Taking \(x_N = 0\) and \(x_B = B^{-1} b\) is the basic solution corresponding to the basis \(B\). It is the basic feasible solution (BFS) corresponding to \(B\) if \(B^{-1} b \geq 0\).
Tableau form

Assume we have a BFS, so $B^{-1}b \geq 0$. The vector of reduced costs is

$$\tilde{c} := c_N - N^T B^{-T} c_B \in \mathbb{R}^{n-m}$$

The LP can be written equivalently as a simplex tableau:

$$\begin{align*}
\text{min} & \quad c_B^T B^{-1} b + \tilde{c}^T x_N \\
\text{subject to} & \quad x_B + B^{-1} N x_N = B^{-1} b \\
& \quad x_B, x_N \geq 0.
\end{align*}$$
If all the reduced costs are nonnegative then the LP is in **optimal form**:

**Theorem**

*If the vector of reduced costs \( \tilde{c} \) is nonnegative then the BFS \( x_N = 0, x_B = B^{-1}b \) is optimal.*

**Proof.**

Since \( \tilde{c} \geq 0 \) and \( x_N \geq 0 \), any feasible solution has value at least \( c_B^T B^{-1}b \). This value is attained by the BFS.
Optimal form

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Finding a better solution

If some component $\tilde{c}_k < 0$ then we can define a simplex direction: we try to increase $x_k$ while keeping the other nonbasic variables equal to zero and compensating using the basic variables.

Let $R$ denote the current set of nonbasic variables. Let $\tilde{a}^r$ denote the column of $B^{-1}N$ corresponding to nonbasic variable $r \in R$. 
Simplex direction

The LP can be written equivalently as

$$\begin{align*}
\text{min} & \quad c_B^T B^{-1} b + \tilde{c}_k x_k + \sum_{r \in R \setminus k} \tilde{c}_r x_r \\
\text{subject to} & \quad x_B + \tilde{\alpha}^k x_k + \sum_{r \in R \setminus k} \tilde{\alpha}^r x_r = B^{-1} b \\
& \quad x_B, x_k, \quad \{x_r : r \in R \setminus k\} \geq 0.
\end{align*}$$

If we set $x_k = t$ and $x_r = 0$ for $r \in R \setminus k$ then we must set $x_B = B^{-1} b - \tilde{\alpha}^k t$. In vector form, the modified point is

$$\begin{bmatrix}
x_B \\
x_k \\
x_{r \in R \setminus k}
\end{bmatrix} = \begin{bmatrix}
B^{-1} b - \tilde{\alpha}^k t \\
t \\
0
\end{bmatrix} = \begin{bmatrix}
B^{-1} b \\
0 \\
0
\end{bmatrix} + t \begin{bmatrix}
-\tilde{\alpha}^k \\
1 \\
0
\end{bmatrix}. \quad (2)$$

This point is feasible for $t \geq 0$ for which $x_B$ remains nonnegative.
Unbounded form

If $\tilde{a}^k \leq 0$ then the tableau is in **unbounded form**:

**Theorem**

*If there exists a nonbasic variable $x_k$ with $\tilde{c}_k < 0$ and $\tilde{a}^k \leq 0$ then the LP has an unbounded optimal value.*

**Proof.**

Under the conditions of the theorem, the point given by (2) is feasible for any $t \geq 0$. Further, it has objective function value $c_B^T B^{-1} b + \tilde{c}_k t \to -\infty$ as $t \to \infty$, since $\tilde{c}_k < 0$. 
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Pivoting

If $\tilde{a}^k$ has at least one positive component then $x_k$ enters the basis and some other variable leaves the basis. The leaving variable $j$ is chosen using a minimum ratio test:

$$j \in \arg\min \left\{ \frac{(B^{-1}b)_i}{\tilde{a}^k_i} : i \in B, \tilde{a}^k_i > 0 \right\},$$

let

$$\bar{t} := \min \left\{ \frac{(B^{-1}b)_i}{\tilde{a}^k_i} : i \in B, \tilde{a}^k_i > 0 \right\}.$$

Note from (2) that taking $t = \bar{t}$ gives $x_B \geq 0$, with $x_j = 0$ (and possibly other $x_{i \in B} = 0$ also.)
The updated solution is a BFS.

Proof.

(sketch) The leaving column is a linear combination of $a^k$ and the remaining basic columns, since otherwise we would have had $\bar{a}_j^k = 0$.

The update from one BFS to the next is a pivot. The updated BFS is adjacent to the original one, and they are joined by an edge of the feasible region.
Get another BFS

**Theorem**

*The updated solution is a BFS.*

**Proof.**

(sketch) The leaving column is a linear combination of $a^k$ and the remaining basic columns, since otherwise we would have had $	ilde{a}^k_j = 0$.

The update from one BFS to the next is a **pivot**. The updated BFS is **adjacent** to the original one, and they are joined by an **edge** of the feasible region.