Name:

# MATP6620/ISYE6770

# **Combinatorial Optimization and Integer Programming**

Spring 2015

Midterm Exam, Friday, April 24, 2015.

Please do all four problems. Show all work. No books or calculators allowed. You may use any result from class, the homeworks, or the texts, except where stated. You may use one sheet of handwritten notes. The exam lasts 110 minutes.

SOLUTIONS

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|---|---|---|
| Q1 | | /20 |
| Q2 | | /35 |
| Q3 | | /20 |
| Q4 | | /25 |
| Total | | /100 |

1. We want to find a maximum weighted clique on the graph $G = (V, E)$. Let $n = |V|$. We define binary variables $x \in \mathbb{B}^n$ by

$$x_v = \begin{cases} 1 & \text{if vertex } v \text{ is in the clique} \\ 0 & \text{otherwise} \end{cases}$$

and we denote the set of feasible vectors $x$ by $S$.

(a) (10 points) Show that $\dim(S) = n$.

(b) (10 points) An independent subset $U \subseteq V$ is a collection of vertices, no two of which are adjacent in $G$. Any clique contains at most one of the vertices in $U$. Show that the corresponding inequality

$$\sum_{v \in U} x_v \leq 1$$

defines a facet of the convex hull of $S$ if $U$ is a maximal ~~clique~~ independent set

(a) Origin and all unit vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$ are feasible.
So $\dim(S) = n$.

(b) We give $n$ linearly independent vectors that satisfy the constraint at equality:

(i) For each $v \in U$, take $e_v$, the corresponding unit vector.

(ii) For each $w \notin U$, $\exists\, v \in U$ that is adjacent to $w$, since $U$ is maximal.
Then $x = e_v + e_w$ is in $S$.

These $n$ vectors are linearly independent, so they are affinely independent, so we have a facet.

2. The nonnegative integer variables $x \in \mathbb{Z}^4$ satisfy the linear equality

$$x_1 + 1.6x_2 - 2.7x_3 + 0.1x_4 = 2.5. \quad (1)$$

(a) (7 points) Give a linear inequality of the form $g_2x_2 + g_3x_3 + g_4x_4 \geq 1$ that must be satisfied by $\{x_2, x_3, x_4\}$ if $x_1 \leq 2$.

(b) (8 points) Give a linear inequality of the form $h_2x_2 + h_3x_3 + h_4x_4 \geq 1$ that must be satisfied by $\{x_2, x_3, x_4\}$ if $x_1 \geq 3$.

(c) (5 points) Argue why the inequality

$$\max\{g_2, h_2\}x_2 + \max\{g_3, h_3\}x_3 + \max\{g_4, h_4\}x_4 \geq 1$$

is valid for any nonnegative integer solution to (1). Write down the inequality.

(d) We can define a new variable $y_1$ so that

$$y_1 - 0.4x_2 + 0.3x_3 + 0.1x_4 = 0.5. \quad (2)$$

i. (5 points) Show that $y_1$ must be integral.

ii. (10 points) Derive another valid inequality on $x_2$, $x_3$, and $x_4$, using the disjunction that either $y_1 \leq 0$ or $y_1 \geq 1$, as in parts 2a–2c. Which valid inequality is stronger?

(a) If don't exploit integrality: $1.6x_2 - 2.7x_3 + 0.1x_4 \geq 0.5$ $(\alpha)$

or $3.2x_2 - 5.4x_3 + 0.2x_4 \geq 1$

If exploit integrality: $0.6x_2 + 0.3x_3 + 0.1x_4 \geq 0.5$ $(\beta)$

or $1.2x_2 + 0.6x_3 + 0.2x_4 \geq 1$

(b) If don't exploit integrality: $1.6x_2 - 2.7x_3 + 0.1x_4 \leq -0.5$

or $-3.2x_2 + 5.4x_3 - 0.2x_4 \geq 1$ $(\gamma)$

If exploit integrality: $-1.6x_2 + 2.7x_3 - 0.1x_4 = 0.5 +$ nonnegative integer

$\Rightarrow 0.4x_2 + 0.7x_3 + 0.9x_4 \geq 0.5$ $(\delta)$

or $0.8x_2 + 1.4x_3 + 1.8x_4 \geq 1$

(c) ~~The~~ The combined inequality is weaker than the earlier ones, so it is valid.

If don't exploit integrality: $3.2x_2 + 5.4x_3 + 0.2x_4 \geq 1$

If exploit integrality: $1.2x_2 + 1.4x_3 + 1.8x_4 \geq 1$

(intentionally left blank)

(d) (i)

$$y_1 = 0.5 + 0.4x_2 - 0.3x_3 - 0.1x_4$$

$$= 0.5 - 1.6x_2 + 2.7x_3 - 0.1x_4 + 2x_2 - 3x_3$$

$$= x_1 - 2 + 2x_2 - 3x_3,$$

so integral

(ii) $y_1 \leq 0$: Need $-0.4x_2 + 0.3x_3 + 0.1x_4 \geq 0.5$

or $-0.8x_2 + 0.6x_3 + 0.2x_4 \geq 1$ $(\eta)$

$y_1 \geq 1$: Need $-0.4x_2 + 0.3x_3 + 0.1x_4 \leq -0.5$

or $0.8x_2 - 0.6x_3 - 0.2x_4 \geq 1$ $(\xi)$

Combining gives the valid inequality:

$$0.8x_2 + 0.6x_3 + 0.2x_4 \geq 1$$

3. In this problem, we consider two problems in $\mathcal{NP}$:

whose sum $\sum_{j=1}^{n} a_j$ is even

**The partition problem:** Given $n$ positive integers $\{a_1, \ldots, a_n\}$, choose a subset $J \subseteq \{1, \ldots, n\}$ such that

$$\sum_{j \in J} a_j = \sum_{j \notin J} a_j.$$

**2-machine scheduling:** Given two identical machines, $m$ jobs requiring positive integer processing time $b_j$ for $j = 1, \ldots, m$, and a positive integer $T$, can the jobs be scheduled so that all the jobs are finished by time $T$? Note that each job must be processed on exactly one machine, each machine can only process one job at a time, once a job is started it must be run to completion, and the jobs can be processed in any order.

(a) (10 points) Show that any instance of the partition problem can be polynomially transformed into an equivalent instance of the 2-machine scheduling problem.

(b) (5 points) Assume the 2-machine scheduling problem is $\mathcal{NP}$-Complete. Can we conclude that the partition problem is $\mathcal{NP}$-Complete?

(c) (5 points) Assume the partition problem is $\mathcal{NP}$-Complete. Can we conclude that the 2-machine scheduling problem is $\mathcal{NP}$-Complete?

(a) Construct a scheduling instance by taking $n$ jobs with processing times $a_j$, and setting $T = \frac{1}{2}\sum_{j=1}^{n} a_j$.
The 2-machine scheduling problem is feasible $\Leftrightarrow$ the partition problem is feasible.

(b) No: We've only looked at very specialized instances of 2-machine scheduling.

(c) Yes: this is the correct order for a reduction.

4. Let $k \geq 2$ and $p \geq 2$ be positive integers. Assume we have $n = kp$ items. We want to cluster the items into $k$ clusters $C_1, \ldots, C_k$ each containing exactly $p$ clusters. We can represent the assignment of items to clusters using a $n \times k$ matrix $Y$ with

$$Y_{ir} = \begin{cases} 1 & \text{if item } i \text{ is in cluster } C_r \\ 0 & \text{otherwise.} \end{cases}$$

Let $e^q$ represent the $q$-dimensional vector of ones.

(a) (10 points) Show that $Ye^k = e^n$ and $Y^T e^n = pe^k$.

(b) To set up a semidefinite programming relaxation of the clustering problem, we can define an $n \times n$ matrix $X = YY^T$.

i. (10 points) Show each diagonal entry of $X$ is equal to one.

ii. (5 points) What is the value of the product $Xe^n$?

(a) ~~X~~ For any $i \in \{1, \ldots, n\}$, we have

$$(Ye^k)_i = \sum_{r=1}^{k} Y_{ir} e^k_r = \sum_{r=1}^{k} Y_{ir} = 1$$

since each $i$ is in exactly one cluster

For any $r \in \{1, \ldots, k\}$, we have

$$(Y^T e^n)_r = \sum_{j=1}^{n} Y_{jr} e^n_r = \sum_{j=1}^{n} Y_{jr} = p,$$

since this adds up the number of entries in cluster $r$.

(b) (i) $$X_{ii} = \sum_{r=1}^{k} Y_{ir} Y_{ir} = \sum_{r=1}^{k} (Y_{ir})^2 = 1$$

$\because$ $Y_{ir} = 1$ if item $i$ in cluster $r$, and $0$ otherwise, so it is in exactly one cluster.

(ii) $$Xe^n = YY^T e^n = p\,Ye^k = pe^n.$$