Name:

MATP6620/ISYE6770

Combinatorial Optimization and Integer Programming

Spring 2013

Midterm Exam, Friday, April 19, 2013.

Please do all four problems. Show all work. No books or calculators allowed. You may use any result from class, the homeworks, or the texts, except where stated. You may use one sheet of handwritten notes. The exam lasts 110 minutes.

SOLUTION.

	Mean	Med	St D	
Q1	24	29	7.35	/30
Q2	15	18	5.28	/20
Q3	22	21	5.49	/30
Q4	17	18	3.40	/20
Total				/100

Mean 77.84

St Dev 13.14

Median 81

Min 48

Max 98

1. (30 points; each part is worth 10 points)

   (a) The binary variables x_1, x_2, x_3, x_4, and x_5 must satisfy the knapsack constraint

   $$3x_1 + 4x_2 + 5x_3 + 6x_4 + 4x_5 \leq 11. \quad (1)$$

   Show that the valid cover inequality

   $$x_1 + x_2 + x_3 \leq 2$$

   has Chvatal rank equal to one. (Note: You will need to use the valid inequalities $x_i \leq 1$ in your derivation.)

   (b) The binary variables x_i, $i = 1, \ldots, n$ must satisfy the knapsack constraint

   $$\sum_{i=1}^{n} a_i x_i \leq b \qquad (2)$$

   where b and a_i, $i = 1, \ldots, n$ are positive scalars. Assume $b > a_i$ for $i = 1, \ldots, n$. Assume further that there exists a subset $J \subseteq \{1, \ldots, n\}$ with

   $$\sum_{i \in J} a_i > b. \qquad (3)$$

   i. Give a valid cover inequality constraint.
   ii. Show that your cover inequality has Chvatal rank equal to one.

(a) Add $(1) + 4(x_1 \leq 1) + 3(x_2 \leq 1) + 2(x_3 \leq 1)$:

$$7x_1 + 7x_2 + 7x_3 + 6x_4 + 4x_5 \leq 11 + 9 = 20$$

Divide by 7:

$$x_1 + x_2 + x_3 + \frac{6}{7}x_4 + \frac{4}{7}x_5 \leq 2\tfrac{6}{7}$$

Round down:

$$x_1 + x_2 + x_3 \leq 2, \text{ as required.}$$

(b) (i) $\sum_{i \in J} x_i \leq |J| - 1$.

(ii) Add $(2) + \sum_{i \in J} \left[(\bar{a} + 1 - a_i) x_i \leq 1 \right]$ where $\bar{a} = \max_{i=1,\ldots,n} \{a_i\}$:

$$\sum_{i \in J} (\bar{a}+1) x_i + \sum_{i \notin J} a_i x_i \leq b + \sum_{i \in J} (\bar{a} + 1 - a_i) = (\bar{a}+1)|J| + b - \sum_{i \in J} a_i$$

Divide by $\bar{a}+1$ and round down:

$$\sum_{i \in J} x_i \leq |J| + \left\lfloor b - \sum_{i \in J} a_i \right\rfloor \leq |J| - 1, \text{ from } (3).$$

2. (20 points)

   The NODE PACKING and MAX CLIQUE feasibility problems can be described as follows:

   > NODE PACKING: Given a graph $G = (V, E)$ and a integer k, does there exist a subset $U \subseteq V$ with $|U| \geq k$ where no two of the vertices in U share an edge?
   > MAX CLIQUE: Given a graph $G = (V, E)$ and an integer p, does there exist a clique $W \subseteq V$ with with $|W| \geq p$?

   Using the fact that NODE PACKING *is NP-Complete*, show that the MAX CLIQUE problem is NP-complete.

Max Clique is in NP: can check whether a set of vertices is a clique, and has right cardinality.

To reduce node packing to clique:
Given an instance of Node Packing, on $G=(V,E)$, construct an instance of ~~an~~ Max Clique with graph $H = (W, F)$ where:

$W = V$

$F =$ complement of E, so $e \in F \Leftrightarrow e \notin E$.

Packings in G correspond exactly to cliques in H.

3. (30 points; each part is worth 10 points)

   (a) Assume the nonnegative scalar integer variable y and the nonnegative scalar continuous variable x satisfy the inequality

   $$x + y \geq b \qquad (4)$$

   where b is a non-integral scalar parameter. Let f be the fractional part of b, so $b = \lfloor b \rfloor + f$ and $0 < f < 1$. Prove that the inequality

   $$\frac{1}{f}x + y \geq \lceil b \rceil \qquad (5)$$

   is valid.

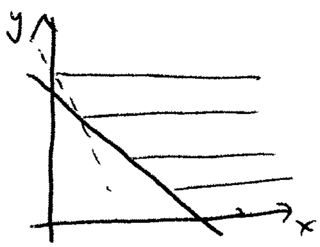
   (b) The optimal solution of the LP relaxation of the mixed integer program

   $$\begin{array}{llllllllll} \min & 2x_1 & + & 5x_2 & + & y_1 & + & y_2 & & \\ \text{subject to} & x_1 & & & + & y_1 & & & \geq & 2.5 \\ & x_1 & + & 2x_2 & + & 3y_1 & + & y_2 & \geq & 8.5 \\ & & & 3x_2 & & & + & y_2 & \geq & 1.2 \\ & x_i & \geq & 0, & i = 1,2 & & & y_i & \geq & 0, \text{ integral}, i = 1,2 \end{array}$$

   is $x_1 = x_2 = 0$, $y_1 = 2.5$, $y_2 = 1.2$.

   i. Give three valid inequalities of the type in part 3a that are violated by the solution to the LP relaxation.
   ii. Show that none of your inequalities is implied by the other two.

(a)



Clearly valid if $y \geq \lceil b \rceil$.
If $y = \lfloor b \rfloor$: need $x \geq f$ from (4), so (5) satisfied.
If $y < \lfloor b \rfloor$: need $x \geq f + \lfloor b \rfloor - y$, so $\frac{x}{f} + y \geq 1 + \lfloor b \rfloor$.

(b) (i) $2x_1 + y_1 \geq 3$ (6) $\quad 2x_1 + 4x_2 + 3y_1 + y_2 \geq 9$ (7) $\quad 15x_2 + y_2 \geq 2$ (8).

(8) not implied by others: $x = (0,0)$, $y = (3, 1.2)$, satisfies all constraints except (8)

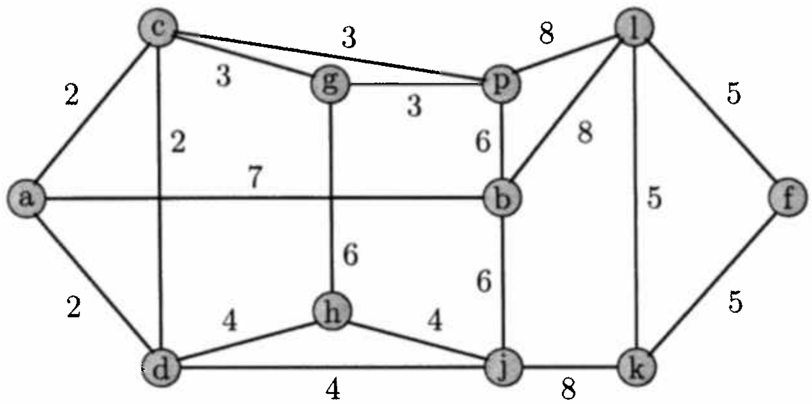
(6) not implied by others: $x = (0,0)$, $y = (2.5, 2)$ satisfies all constraints except (6). ~~satisfies all constraints except (7).~~

(7) not implied by others: ~~[illegible]~~ ~~[illegible]~~

$x = (1.5, \frac{2}{15})$, $y = (0,0)$ satisfies (6), (8), but not (7).

4. (20 points)

Show that the shortest Hamiltonian cycle in the following graph has length 50. Make sure to prove your solution is optimal.



Tour: $a \underset{2}{-} d \underset{4}{-} h \underset{4}{-} j \underset{8}{-} k \underset{5}{-} f \underset{5}{-} l \underset{8}{-} b \underset{6}{-} p \underset{3}{-} g \underset{3}{-} c \underset{2}{-} a$

Length: 50, as required.

Can't do better:

Use at most two from $(a,c), (a,d), (c,d)$: length 4
Use at most two from $(d,h), (d,j), (h,j)$: length 8
Use at most two from $(c,g), (c,p), (g,p)$: length 6
Use at most two from $(f,k), (f,l), (k,l)$: length 10

} length 28 from 8 edges.

Need three more edges.

Must use two of $(b,l), (j,k), (l,p)$: length 16

Must use at least one of the remaining edges:
$(b,j), (b,p), (a,b), (g,h)$. length ≥ 6.

So total length is at least $28+16+6 = 50$.