# MATROIDS

Prototypes of independence systems with "nice" properties.

<u>Defn</u> Let $N = \{1, \ldots, n\}$ be a finite set, and let $\mathcal{F}$ be a set of subsets of $N$. $I = (N, \mathcal{F})$ is an INDEPENDENCE SYSTEM if $F_1 \in \mathcal{F}$, $F_2 \subseteq F_1 \Rightarrow F_2 \in \mathcal{F}$. Elements of $\mathcal{F}$ are called INDEPENDENT SETS, and the remaining subsets of $N$ are called DEPENDENT SETS.

Eg: (i) Acyclic subgraphs
(ii) Linearly independent vectors (in homework)
(iii) Matchings.

<u>Defn</u> Given an independence system $I = (N, \mathcal{F})$, we say that $F_i$ is a <u>maximal</u> independent set if $F_i \cup \{j\} \notin \mathcal{F}$ for any $j \in N \setminus F_i$. A maximal independent set $T$ is <u>maximum</u> if $|S| \leq |T|$ for all $S$ in $\mathcal{F}$.

<u>Defn</u> $M = (N, \mathcal{F})$ is a MATROID if $M$ is an independence system in which for any subset $T \subseteq N$, every independent set in $T$ that is maximal in $T$ has the same cardinality.

omit, since on HW6

Eg: (i), (ii) YES (i) is so-called GRAPHIC MATROID.

(iii) NO

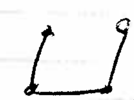
(iv) Uniform matroids: Every subset of size $\leq k$ is independent.

(v) Partition matroids: on disjoint finite sets $E_i$.

$E = \bigcup E_i$

$F \subseteq E$ indep if $|F \cap E_i| \leq 1$ for each $i$.

Eg: on ~~graph~~ digraph: $E_i$ = set of edges entering vertex $i$.

Eg: Need maximal = maximum "FOR EVERY SUBSET" because, eg matching on $C_4$ vs matching on the subset 

Associate weights with elements of base set $N$

Theorem. Greedy algorithm solves the matroid problem

~~$\max \{ \sum_{i=1} c_i x_i : x$ is incidence vector of independent set$\}$.~~

$\max \{ \sum_{j \in S} c_j : S$ indep set$\}$. Eg spanning tree

Theorem If $(N, F)$ is an indep. sys but not a matroid then there exists a weight function for which greedy fails.

Eg. matching

Theorem Let $r(T)$ be the size of the largest independent set in $T \subseteq N$.

Then $r(T) + r(S) \geq r(T \cup S) + r(T \cap S)$. (~~Submod func~~ A function satisfying this condition is SUBMODULAR.)

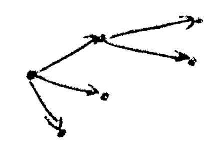
Eg. Spanning tree.

## k-matroid intersection theorem

Given k matroids $M_i = (N, F_i)^{i=1,\dots,k}$ on base set $N$, and weight vector $c_i$, $i \in N$, solve

$$\max_S \left\{ \sum_{j \in S} c_j : S \in \bigcap_{i=1}^{k} F_i \right\}$$

Application for $k=2$: find a branching in a digraph, i.e., a spanning tree where at most one edge enters each vertex



Solvable in poly time for $k=2$.

NP-hard for $k=3$.

Reduction from HAMILTONIAN PATH IN DIRECTED GRAPH.

## Dual Matroids:

[Call the independent sets of maximum cardinality the BASES OF A MATROID.]

Then all the complements of the bases form ~~another~~ the bases of another matroid: the dual matroid

Eg: • Uniform matroid. Dual matroid is also uniform.

• Partition matroids. ~~Ind~~ Independent provided don't take all of a set $C_i$

• Acyclic subgraphs. Bases are complements of spanning trees

[Minimal dependent sets are circuits of a matroid.]

Dual ~~*~~ independent sets: sets of edges that do not disconnect the graph. COCYCLE MATROID

Why does greedy work?

Assume there is a better soln.

There is something in this better soln that is not in the greedy soln.

Find the smallest dependent set when we add this extra element to the greedy soln.

Remove ~~the part~~ worst element from the dependent set, giving a new indep set, better than greedy.

Greedy should have found this instead of the one it did.