LP Relaxations of Mixed Integer Programs

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LP relaxations

We want to solve an integer program by solving a sequence of linear programs. We can relax the integrality constraints to give an **LP relaxation**. Let

$$\begin{aligned} S &:= \{x \in \mathbb{Z}^n : Ax \geq b,\, x \geq 0\} \\ &= \mathbb{Z}^n \cap P \\ \text{where } P &:= \{x \in \mathbb{R}^n : Ax \geq b,\, x \geq 0\}. \end{aligned}$$

Since we restrict $x \geq 0$, the polyhedron P has extreme points if it is nonempty.

Convex hull of S

Theorem

The convex hull of S is a polyhedron.

The proof is in Nemhauser and Wolsey.

It is important to note that the extreme rays of conv(S) are the extreme rays of P.

Equivalent LP formulation

Our integer program of interest is

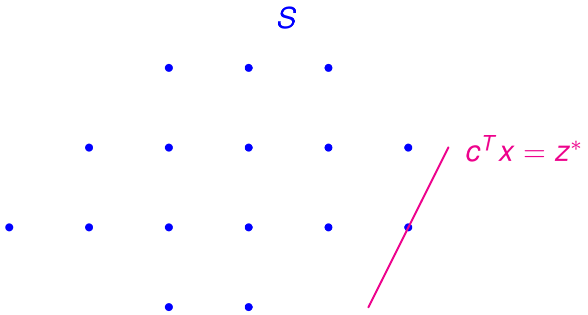
$$\min\{c^T x \,:\, x \in S\} \qquad (IP).$$

We relate this to the linear program

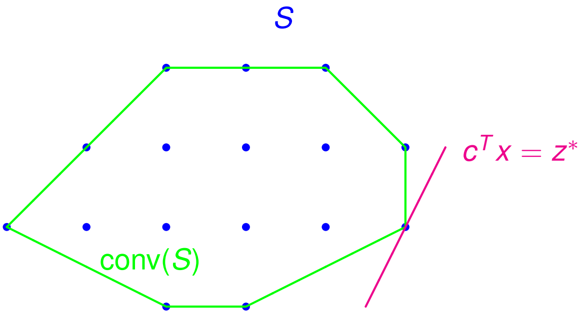
$$\min\{c^T x \,:\, x \in \text{conv}(S)\} \qquad (CIP).$$

We don't know the constraints of the linear program (CIP) explicitly.

(IP) and (CIP)



(IP) and (CIP)



Value of (IP) and (CIP)

Theorem

For any $c \in \mathbb{R}^n$, we have:

1. *The objective value of (IP) is unbounded below if and only if the objective value of (CIP) is unbounded below.*
2. *If (CIP) has a bounded optimal value then it has an optimal solution that is an optimal solution to (IP). This optimal solution is an extreme point of conv(S).*
3. *If x^0 is an optimal solution to (IP) then it is an optimal solution to (CIP).*

The proof is in the text. It relies on the fact that the minimizer of a concave function over a convex set is at an extreme point.

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Possible cases for an integer linear program

Corollary

(IP) is either unbounded or infeasible or has an optimal solution.

Set up LP relaxation

So in principle, could solve (IP) by solving the **linear program** (CIP). But we don't know a polyhedral description of conv(S).

Look at the **LP relaxation**

$$\min\{c^Tx \,:\, Ax \geq b,\, x \geq 0\} \quad = \quad \min\{c^Tx \,:\, x \in P\}. \qquad (LP)$$

Compare (IP) to LP relaxation

We can relate (LP) to (IP) in the following theorem. The proof exploits the fact that $S \subseteq P$.

Theorem

Let z^{LP} and z^{IP} be the optimal values of (LP) and (IP) respectively.

1. *If $P = \emptyset$ then $S = \emptyset$.*
2. *If z^{LP} is finite then either $S = \emptyset$ or z^{IP} is finite. If $S \neq \emptyset$ then $z^{IP} \geq z^{LP}$.*
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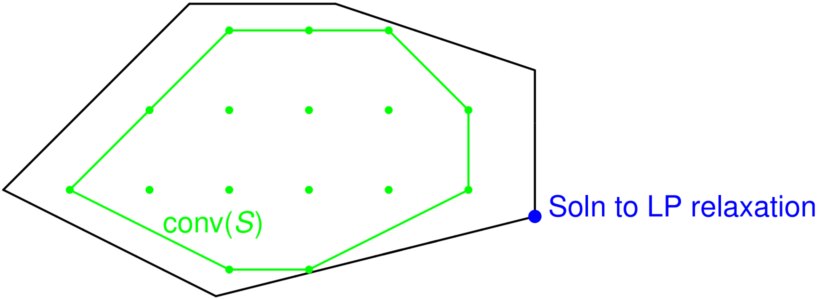
Cutting plane algorithm illustration

Solve linear program relaxation

$\{x \in \mathbb{R}^n_+ : Ax \geq b\}$



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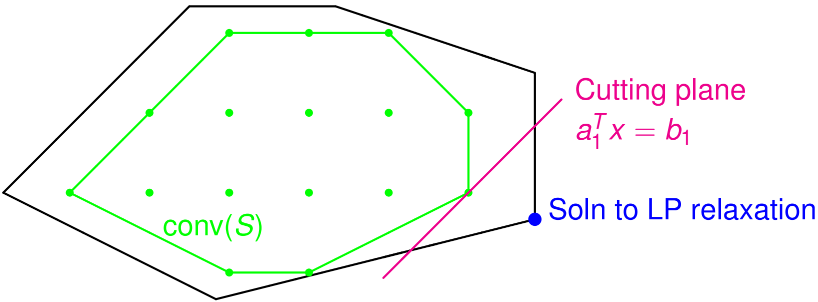
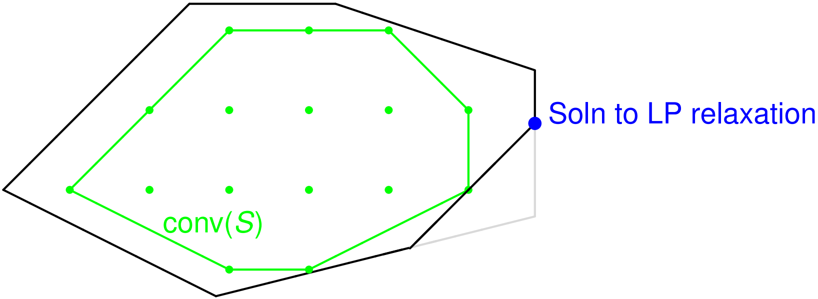


Cutting plane algorithm illustration

Solve linear program relaxation

$\{x \in \mathbb{R}^n_+ : Ax \geq b, \; a_1^T x \geq b_1\}$



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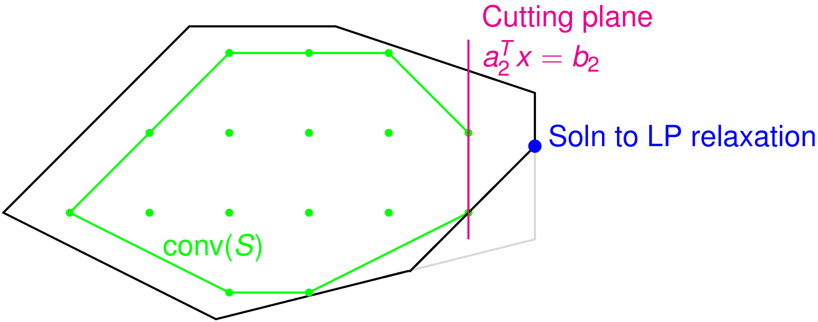

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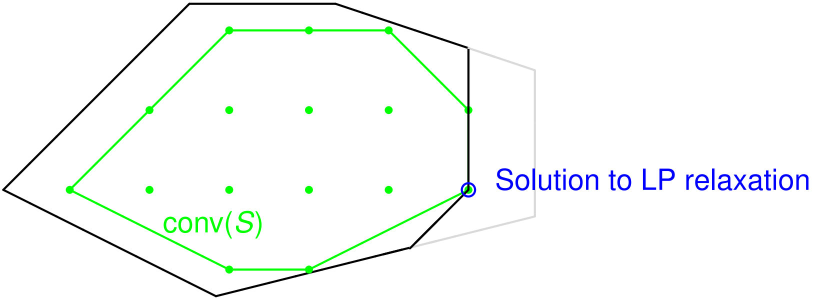

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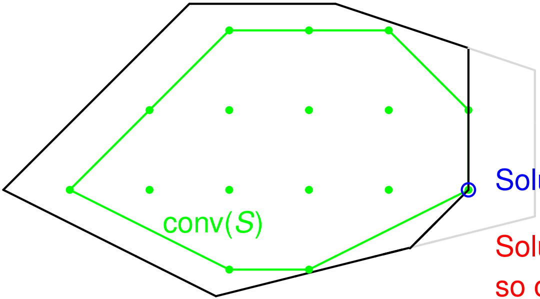

Solution to LP relaxation

Solution is integral, so optimal to IP

A cutting plane approach

An integer program has **many LP relaxations.**

The aim of a cutting plane approach is to add linear constraints to the original LP relaxation in order to **tighten up** the LP relaxation.

These linear constraints are satisfied by every point in S but they might not be satisfied by every point in P.

Adding these constraints gives an **improved LP relaxation** of the integer program.

We would like to improve the relaxation so that it resembles conv(S) in some sense, at least in the neighborhood of an optimal solution to (IP).

Polyhedral description of conv(S)

We would like to get a polyhedral description of conv(S).

We'd like as compact a description as possible.

We first set up some notation.

Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$, we construct the polyhedron

$$Q := \{x \in \mathbb{R}^n : Ax \geq b\}.$$

Note: We have changed notation from Section 1. In particular, any nonnegativity constraints are included in the system $Ax \geq b$.

Example 0

Full-dimensional example

$$Q := \left\{ x \in \mathbb{R}^2 : \begin{array}{rcrcr} x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 6 \\ x_1 & - & x_2 & \leq & 2 \\ 2x_1 & + & x_2 & \leq & 10 \\ x_1 & & & \geq & 0 \\ & & x_2 & \geq & 0 \end{array} \right\}$$

Example 1

First example with implicit equality in $\mathbb{R}^2$

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Example 2

Second example with implicit equality in $\mathbb{R}^2$

$$Q := \left\{ x \in \mathbb{R}^2 \; : \; \begin{array}{rcrcr} 2x_1 & + & x_2 & \leq & 4 \\ x_1 & + & x_2 & \geq & 4 \\ x_1 & & & \geq & 0 \\ & & x_2 & \geq & 0 \end{array} \right\}$$

Implicit equality constraints

We construct three sets of indices:

$$\begin{aligned} M &:= \{1, \ldots, m\} \\ M^{=} &:= \{i \in M \,:\, A_i^T x = b_i \,\forall x \in Q\} \\ M^{\geq} &:= M \setminus M^{=} \\ &= \{i \in M \,:\, A_i^T x > b_i \text{ for some } x \in Q\}. \end{aligned}$$

We also use the notation $A^{=}$, $A^{\geq}$, $b^{=}$ and $b^{\geq}$ in an analogous way.

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Valid inequalities

Definition

The inequality $\pi^T x \geq \pi_0$ is a valid inequality for Q if it is satisfied by all points in Q. We also represent the inequality using the pair (π, π_0).

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Let $\pi^T x \geq \pi_0$ be a valid inequality for Q. Let $H = \{x \in \mathbb{R}^n : \pi^T x = \pi_0\}$ so $F := Q \cap H$ is a face of Q. Then (π, π_0) represents F. If $F \neq \emptyset$ then (π, π_0) supports Q.

We extend the notation $M^=$, $M^\geq$ to the face F using the notation $M_F^=$, $M_F^\geq$.

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Impicit equalities for faces

Proposition

If F is a nonempty face of Q then F is a polyhedron and

$$F = \{x \in \mathbb{R}^n : a_i^T x = b_i \, \forall i \in M_F^=, \, a_i^T x \geq b_i \, \forall i \in M_F^\geq\},$$

where $M_F^= \supseteq M^=$ and $M_F^\geq \subseteq M^\geq$.

Facets

Proposition

If F is a facet of Q then there exists some inequality $a_k^T x \geq b_k$ with $k \in M^{\geq}$ that represents F.

Proof.

Now dim(F) = dim(P)-1, so rank$(A_F^=, b_F^=) = \text{rank}(A^=, b^=) + 1$.
Therefore, we only need one inequality in $M^{\geq}$ that is in $M_F^=$, and this inequality represents the facet. □

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One inequality per facet

Proposition

For each facet F of Q, one of the inequalities representing F is necessary in the description of Q.

Proposition

Every inequality $a_i^T x \geq b_i$ for $i \in M^{\geq}$ that represents a face of Q of dimension less than $\dim(Q) - 1$ is irrelevant to the description of Q.

Thus, later we will emphasize finding **facets of the convex hull** of S.

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Minimal representation of polyhedron

With these propositions, we can now state two theorems showing how a polyhedron can be minimally represented using linear equalities and inequalities.

Theorem (full dimensional case)

A full dimensional polyhedron Q has a unique (to within scalar multiplication) minimal representation by a finite set of linear inequalities. In particular, for each facet F_i of Q with $i = 1, \ldots, t$, there is an inequality $a_i^T x \geq b_i$ (unique to within scalar multiplication) representing F_i and

$$Q = \{x \in \mathbb{R}^n : a_i^T x \geq b_i, \ i = 1, \ldots, t\}.$$

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Minimal representation of polyhedron

Theorem (not full dimensional)

If $\dim(Q) = n - k$ with $k > 0$ then let t be the number of facets F_i of Q, and we have

$$Q = \{x \in \mathbb{R}^n : a_i^T x = b_i \text{ for } i = 1, \ldots, k, \\ a_i^T x \geq b_i \text{ for } i = k+1, \ldots, k+t\}.$$

The set $\{(a_i, b_i) : i = 1, \ldots, k\}$ is a maximal set of linearly independent rows of $(A^=, b^=)$.
For $i = k+1, \ldots, k+t$, (a_i, b_i) is any inequality from the equivalence class of inequalities representing the facet F_i of Q.

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Summary

The aim in a cutting plane approach is to try to develop a **minimal description of the polyhedron** conv(S), at least in the neighborhood of an optimal point to (IP).

This pair of theorems tells us that it is enough to determine all the facets of conv(S), as well as a description of the affine hull of conv(S).