1 Rounding

So far, we’ve considered problems where the variables are continuous. We now look at integer optimization problems, where (some of) the variables are constrained to take integer values.

We might try ignoring the integrality restriction, solving the resulting linear optimization problem relaxation, and rounding the solution. But this may not work well, as in the following example:

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad -x_1 - x_2 \\
\text{subject to} & \quad -2x_1 + 4x_2 \leq 1 \\
& \quad 9x_1 - 16x_2 \leq 4 \\
& \quad x_1, x_2 \text{ integer} \\
& \quad x_1, x_2 \geq 0 
\end{align*}
\]

The optimal solution to the relaxation is \(x = (8, 4.25)\) with value \(z_{LP} = -12.25\). Rounding this gives \(x = (8, 4)\), which is far from feasible, and far from the optimal integral solution, \(x = (4, 2)\), which has value \(-6\).

Note that the value of the relaxation does give a lower bound on the optimal value of the integer optimization problem.

We could try exhaustive enumeration to solve the integer optimization problem, where we check every possible solution. Here, we could check \(x_1 = 0, \ldots, 8\) with \(x_2 = 0, \ldots, 5\), so \(9 \times 6 = 54\) possibilities. We’d check each point for feasibility and if it’s feasible then calculate its objective function value. This is impractical in higher dimensions. For example, if we had ten variables each with 6 possible values then the number of possible points is \(6^{10} = 60,466,176\).
2 An example of Branch-and-bound

As an alternative to exhaustive enumeration, we look at *implicit enumeration*, where we divide up the feasible region and use logical implications to shrink the search space. For example, consider the problem

\[
\begin{align*}
\text{min} & \quad -3x_1 - 2x_2 \\
\text{subject to} & \quad x_1 + x_2 \leq 6 \\
& \quad -x_1 + x_2 \leq 4 \\
& \quad 2x_1 - x_2 \leq 8 \\
& \quad x_1, x_2 \quad \text{integer} \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

We denote the feasible region to the relaxation by $F$, so

\[F := \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 6, -x_1 + x_2 \leq 4, 2x_1 - x_2 \leq 8, x_1, x_2 \geq 0\}.
\]

The solution to the linear optimization relaxation is $x = (4\frac{2}{3}, 1\frac{1}{3})$ with value $-16\frac{2}{3}$. Any integer point must have either $x_2 \leq 1$ or $x_2 \geq 2$. We can exploit this to **branch** on $x_2$, setting up two new relaxations:

- Require $x_2 \geq 2$:
  \[
  \begin{align*}
  \min_{x \in \mathbb{R}^2} & \quad -3x_1 - 2x_2 \\
  \text{subject to} & \quad x_2 \geq 2 \\
  & \quad x \in F
  \end{align*}
  \]

  The optimal solution to this relaxation is $x = (4, 2)$ with value $-16$. Since the **optimal solution to the relaxation is integral**, this point also is the best integral point satisfying $x_2 \geq 2$. So we don’t need to subdivide further.
• Require $x_2 \leq 1$:

$$\min_{x \in \mathbb{R}^2} -3x_1 - 2x_2$$
subject to

- $x_2 \leq 1$
- $x \in F$

The optimal solution to this relaxation is $x = (4\frac{1}{2}, 1)$ with value $-15\frac{1}{2}$. Since the optimal value of the relaxation is worse than the best known integral feasible point $(4, 2)$, the best integer point in $F$ cannot be optimal for the original problem. So we don’t need to subdivide further.

Thus, the solution to the integer optimization problem is $x = (4, 2)$ with value $-16$.

We subdivided the original problem into two, solved relaxations on the two parts, and exploited logic to obtain the solution to the original problem. We did not need to exhaustively enumerate many combinations of integral values.

Note that the original problem can be written:

$$\min \{-3x_1 - 2x_2 : x \in F, x \text{ integral}\}$$

$$= \min \{\min \{-3x_1 - 2x_2 : x \in F, x_2 \geq 2, x \text{ integral}\},$$

$$\min \{-3x_1 - 2x_2 : x \in F, x_2 \leq 1, x \text{ integral}\}\},$$

subdividing it into two.

3 Branching on $x_1$ first

The solution to the relaxation is $x = (4\frac{2}{3}, 1\frac{1}{3})$. We chose to branch on $x_2$, splitting into $x_2 \leq 1$ or $x_2 \geq 2$. We could instead have branched on $x_1$:

• Require $x_1 \geq 5$:

$$\min_{x \in \mathbb{R}^2} -3x_1 - 2x_2$$
subject to

- $x_1 \geq 5$
- $x \in F$

This linear optimization problem is infeasible. Since the relaxation is infeasible, there is no integral point in $F$ satisfying $x_1 \geq 5$. So we don’t need to subdivide further.

• Require $x_1 \leq 4$:

$$\min_{x \in \mathbb{R}^2} -3x_1 - 2x_2$$
subject to

- $x_1 \leq 4$
- $x \in F$

The optimal solution to this relaxation is $x = (4, 2)$ with value $-16$. Since the optimal solution to the relaxation is integral, this point also is the best integral point satisfying $x_2 \geq 2$. So we don’t need to subdivide further.

Thus, again we find that the solution to the integer optimization problem is $x = (4, 2)$ with value $-16$. 

3
4 Possible outcomes for relaxations

We subdivide the feasible region of an integer optimization problem, using simple lower and upper bounds on the variables. We then solve linear optimization relaxations of the integer optimization problem subject to an appropriate subset of these bound constraints. The solution to the relaxation is then used to infer information about the integer optimization problem when it is subject to these bound constraints. There are four possible outcomes when we solve a relaxation:

- The linear optimization relaxation has an integral optimal solution. In this case, this integer solution also solves the integer version of the problem. If this solution is at least as good as the best solution seen so far then it is a possible optimal solution to the complete integer optimization problem.

- The linear optimization relaxation is infeasible. In this case, the integer version is also infeasible.

- The linear optimization relaxation has optimal value that is bounded below by the best known integer feasible solution. In this case, there is again no need to subdivide this problem, since we will not find an integer point better than the best known point found previously.

- The linear optimization problem has optimal value that is smaller than that of the best known integer feasible solution. In this case, the problem needs to be subdivided further.

When we don’t need to subdivide a problem further, we say it is fathomed or pruned.