Network Simplex

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Outline

1. Introduction
2. Example 1
3. Example 2
Network Simplex

The network simplex algorithm is used to solve minimum cost network flow problems. When some of the edges have capacities then a variable can be nonbasic at either its lower bound or its upper bound.

We have a directed graph $G = (V, E)$. We assume $G$ is connected. Each arc $(i, j) \in E$ has a cost per unit flow of $c_{ij}$ and a capacity of $u_{ij}$ (possibly infinite). Each node $i \in V$ has a net supply $b_i$, so

\[
\begin{aligned}
  b_i &> 0 \quad \text{if } i \text{ is a supply node} \\
  b_i &= 0 \quad \text{if } i \text{ is a transshipment node} \\
  b_i &< 0 \quad \text{if } i \text{ is a demand node}.
\end{aligned}
\]
LP formulation

The problem is formulated as

\[
\min_x \quad \sum_{(i,j) \in E} c_{ij} x_{ij} \\
\text{s.t.} \quad \sum_{(i,j) \in E} x_{ij} - \sum_{(k,i) \in E} x_{ki} = b_i \quad \forall i \in V \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0 \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in E.
\]

The dual problem is

\[
\max_{y,z,w} \sum_{i \in V} b_i y_i - \sum_{(i,j) \in E} u_{ij} w_{ij} \\
\text{s.t.} \quad y_i - y_j + z_{ij} - w_{ij} = c_{ij} \quad \forall (i,j) \in E \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad z_{ij}, w_{ij} \geq 0 \quad \forall (i,j) \in E
\]

If some \( x_{ij} \) is unbounded then \( w_{ij} \equiv 0 \).

**Complementary slackness:**
\[x_{ij} z_{ij} = 0 \text{ and } (u_{ij} - x_{ij}) w_{ij} = 0 \text{ for each edge } (i,j) \in E.\]
Complementary slackness

In a basic feasible solution, the set of basic variables constitutes a spanning tree in $G$. Let $\bar{x}$ be a basic feasible solution to $(P)$.

We use **complementary slackness** to find a dual solution $(\bar{y}, \bar{z}, \bar{w})$. We must have:

$$\bar{y}_i - \bar{y}_j = c_{ij} \quad \text{if} \quad \bar{x}_{ij} \text{ is basic.}$$

This underdetermined system can be solved easily: fix one component of $\bar{y}$, and then the remaining components are determined by a chain reaction. Note that $\bar{z}_{ij} = \bar{w}_{ij} = 0$ for the basic $\bar{x}_{ij}$. 
Dual feasibility

We need to **check dual feasibility** for the nonbasic edges. Break into cases depending on which bound is active:

- If nonbasic $\bar{x}_{ij} = 0$ then $\bar{w}_{ij} = 0$ from complementary slackness, so $\bar{z}_{ij} = c_{ij} - \bar{y}_i + \bar{y}_j$.

- If nonbasic $\bar{x}_{ij} = u_{ij}$ then $\bar{z}_{ij} = 0$ from complementary slackness, so $\bar{w}_{ij} = -c_{ij} + \bar{y}_i - \bar{y}_j$.

If $\bar{w} \geq 0$ and $\bar{z} \geq 0$ then $\bar{x}$ is **optimal**.

Else, **perform a simplex iteration**:

- choose a nonbasic arc with $\bar{z}_{ij} < 0$ or $\bar{w}_{ij} < 0$ to enter the basis,
- construct a cycle using the incoming arc and the basic edges,
- adjust flow around the edges of the cycle,
- use the minimum ratio test to determine the leaving variable,
- and update the basis and $\bar{x}$. 
Dual feasibility

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- If nonbasic $\bar{x}_{ij} = 0$ then $\bar{w}_{ij} = 0$ from complementary slackness, so $\bar{z}_{ij} = c_{ij} - \bar{y}_i + \bar{y}_j$.
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Outline

1. Introduction

2. Example 1

3. Example 2
Example 1
All nodes are transshipment nodes.

triples are \((c_{ij}, u_{ij}, x_{ij})\)

arcs are either basic (B) or nonbasic at upper bound (UB) or lower bound (LB)
Example 1

**Dual solution**

\[ B: y_i - y_j = c_{ij} \]

NB 0: \( w_{ij} = 0, z_{ij} = c_{ij} - y_j + y_j \geq 0 \)

NB UB: \( z_{ij} = 0, w_{ij} = -c_{ij} + y_i - y_j \geq 0 \)

Basic arcs give dual values:

\[
\begin{align*}
\bar{x}_{13} \text{ basic implies } & \bar{y}_1 - \bar{y}_3 = 0 \\
\bar{x}_{24} \text{ basic implies } & \bar{y}_2 - \bar{y}_4 = 0 \\
\bar{x}_{41} \text{ basic implies } & \bar{y}_4 - \bar{y}_1 = -1
\end{align*}
\]

Setting \( \bar{y}_4 = 0 \) then gives

\[ \bar{y}_2 = 0, \bar{y}_1 = 1, \bar{y}_3 = 1. \]

Check dual feasibility for nonbasic arcs:

- \( \bar{x}_{12} = 10 = u_{12} \), so \( \bar{z}_{12} = 0 \). Then \( \bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 1. \)
- \( \bar{x}_{23} = 1 = u_{23} \), so \( \bar{z}_{23} = 0 \). Then \( \bar{w}_{23} = -c_{23} + \bar{y}_2 - \bar{y}_3 = -1. \)
- \( \bar{x}_{34} = 10 = u_{34} \), so \( \bar{z}_{34} = 0 \). Then \( \bar{w}_{34} = -c_{34} + \bar{y}_3 - \bar{y}_4 = 1. \)

Thus, arc \((2, 3)\) enters the basis, decreasing from its upper bound of 1.
Example 1

Dual solution

B: \( y_i - y_j = c_{ij} \)

NB 0: \( w_{ij} = 0, z_{ij} = c_{ij} - y_i + y_j \geq 0 \)

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\text{Setting } \bar{y}_4 = 0 \text{ then gives } \\
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Thus, arc \((2, 3)\) enters the basis, decreasing from its upper bound of 1.
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Thus, arc \((2, 3)\) enters the basis, decreasing from its upper bound of 1.
Find cycle, adjust flow
A cycle is created and flow adjusted around the cycle:

Maximum possible value of $t$ is three, with a tie in the minimum ratio between 3 arcs. We choose to have arc $(2, 3)$ remain nonbasic, now at its lower bound.
Updated BFS

The updated BFS is

(cost $c_{ij}$, capacity $u_{ij}$, flow $x_{ij}$)
Example 1

Dual solution

B: \(y_i - y_j = c_{ij}\)

NB 0: \(w_{ij} = 0, z_{ij} = c_{ij} - y_i + y_j \geq 0\)

NB UB: \(z_{ij} = 0, w_{ij} = -c_{ij} + y_i - y_j \geq 0\)

Since we have the same set of basic variables, the dual variables \(y\) are unchanged:

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\begin{align*}
\bar{x}_{13} \text{ basic implies } & \bar{y}_1 - \bar{y}_3 = 0 \\
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Check dual feasibility for nonbasic arcs:

- \(\bar{x}_{12} = 10 = u_{12}\), so \(\ddot{z}_{12} = 0\). Then \(\ddot{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 1\).
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- \(\bar{x}_{34} = 10 = u_{34}\), so \(\ddot{z}_{34} = 0\). Then \(\ddot{w}_{34} = -c_{34} + \bar{y}_3 - \bar{y}_4 = 1\).

Since we are dual feasible, we are now optimal.
Dual solution

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Setting \( \bar{y}_4 = 0 \) then gives \( \bar{y}_2 = 0, \bar{y}_1 = 1, \bar{y}_3 = 1 \).

Check dual feasibility for nonbasic arcs:

- \( \bar{x}_{12} = 10 = u_{12} \), so \( \bar{z}_{12} = 0 \). Then \( \bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 1 \).
- \( \bar{x}_{23} = 0 \), so \( \bar{w}_{23} = 0 \). Then \( \bar{z}_{23} = c_{23} - \bar{y}_2 + \bar{y}_3 = 1 \).
- \( \bar{x}_{34} = 10 = u_{34} \), so \( \bar{z}_{34} = 0 \). Then \( \bar{w}_{34} = -c_{34} + \bar{y}_3 - \bar{y}_4 = 1 \).

Since we are dual feasible, we are now **optimal**.
Example 1

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- $\bar{x}_{34} = 10 = u_{34}$, so $\bar{z}_{34} = 0$. Then $\bar{w}_{34} = -c_{34} + \bar{y}_3 - \bar{y}_4 = 1$.

Since we are dual feasible, we are now **optimal**.
Outline

1. Introduction

2. Example 1

3. Example 2
Example 2

Supply of 8 at node 1, demand of 8 at node 6, other nodes are transshipment.

triples are \((\text{cost } c_{ij}, \text{capacity } u_{ij}, \text{flow } x_{ij})\)
Dual solution

Basic arcs give dual values:

\[
\begin{align*}
\bar{x}_{13} \text{ basic implies } & \bar{y}_1 - \bar{y}_3 = 3 \\
\bar{x}_{32} \text{ basic implies } & \bar{y}_3 - \bar{y}_2 = 7 \\
\bar{x}_{24} \text{ basic implies } & \bar{y}_2 - \bar{y}_4 = 5 \\
\bar{x}_{35} \text{ basic implies } & \bar{y}_3 - \bar{y}_5 = 7 \\
\bar{x}_{56} \text{ basic implies } & \bar{y}_5 - \bar{y}_6 = 9
\end{align*}
\]

Setting $\bar{y}_6 = 0$ then gives $\bar{y}_5 = 9$, $\bar{y}_3 = 16$, $\bar{y}_2 = 9$, $\bar{y}_1 = 19$, $\bar{y}_4 = 4$.

Check dual feasibility for nonbasic arcs:

- $\bar{x}_{12} = 3 = u_{12}$, so $\bar{z}_{12} = 0$. Then $\bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 8$.
- $\bar{x}_{34} = 3 = u_{34}$, so $\bar{z}_{34} = 0$. Then $\bar{w}_{34} = -c_{34} + \bar{y}_3 - \bar{y}_4 = 7$.
- $\bar{x}_{45} = 2 = u_{45}$, so $\bar{z}_{45} = 0$. Then $\bar{w}_{45} = -c_{45} + \bar{y}_4 - \bar{y}_5 = -8$.
- $\bar{x}_{46} = 5 = u_{46}$, so $\bar{z}_{46} = 0$. Then $\bar{w}_{46} = -c_{46} + \bar{y}_4 - \bar{y}_6 = -3$.

Thus, arc $(4, 5)$ enters the basis, decreasing from its upper bound of 2.
Dual solution

Basic arcs give dual values:

\[ \bar{x}_{13} \text{ basic implies } \bar{y}_1 - \bar{y}_3 = 3 \]
\[ \bar{x}_{32} \text{ basic implies } \bar{y}_3 - \bar{y}_2 = 7 \]
\[ \bar{x}_{24} \text{ basic implies } \bar{y}_2 - \bar{y}_4 = 5 \]
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Check dual feasibility for nonbasic arcs:

- \( \bar{x}_{12} = 3 = u_{12}, \) so \( \bar{z}_{12} = 0. \) Then \( \bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 8. \)
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- \( \bar{x}_{45} = 2 = u_{45}, \) so \( \bar{z}_{45} = 0. \) Then \( \bar{w}_{45} = -c_{45} + \bar{y}_4 - \bar{y}_5 = -8. \)
- \( \bar{x}_{46} = 5 = u_{46}, \) so \( \bar{z}_{46} = 0. \) Then \( \bar{w}_{46} = -c_{46} + \bar{y}_4 - \bar{y}_6 = -3. \)

Thus, \textbf{arc (4, 5) enters the basis}, decreasing from its upper bound of 2.
Example 2

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Setting $\bar{y}_6 = 0$ then gives $\bar{y}_5 = 9$, $\bar{y}_3 = 16$, $\bar{y}_2 = 9$, $\bar{y}_1 = 19$, $\bar{y}_4 = 4$.

Check dual feasibility for nonbasic arcs:

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- $\bar{x}_{46} = 5 = u_{46}$, so $\bar{z}_{46} = 0$. Then $\bar{w}_{46} = -c_{46} + \bar{y}_4 - \bar{y}_6 = -3$.

Thus, arc (4, 5) enters the basis, decreasing from its upper bound of 2.
**Example 2**

**Dual solution**

B: \( y_i - y_j = c_{ij} \)

NB 0: \( w_{ij} = 0, z_{ij} = c_{ij} - y_i + y_j \geq 0 \)

NB UB: \( z_{ij} = 0, w_{ij} = -c_{ij} + y_i - y_j \geq 0 \)

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Setting \( \bar{y}_6 = 0 \) then gives

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\bar{y}_5 &= 9, \quad \bar{y}_3 = 16, \quad \bar{y}_2 = 9, \\
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\end{align*}\]

Check dual feasibility for nonbasic arcs:

- \( \bar{x}_{12} = 3 = u_{12} \), so \( \bar{z}_{12} = 0 \). Then \( \bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 8 \).
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Thus, **arc (4, 5) enters the basis**, decreasing from its upper bound of 2.
Find cycle, adjust flow
A cycle is created and flow adjusted around the cycle:

triples are \((c_{ij}, u_{ij}, x_{ij})\)

Maximum possible value of \(t\) is 1, with arc \((3, 2)\) leaving the basis.
Updated BFS

The updated BFS is:

triples are (cost $c_{ij}$, capacity $u_{ij}$, flow $x_{ij}$)
Example 2

Dual solution

Basic arcs give dual values:

\[
\begin{align*}
\bar{x}_{13} \text{ basic implies } & \bar{y}_1 - \bar{y}_3 = 3 \\
\bar{x}_{24} \text{ basic implies } & \bar{y}_2 - \bar{y}_4 = 5 \\
\bar{x}_{35} \text{ basic implies } & \bar{y}_3 - \bar{y}_5 = 7 \\
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\end{align*}
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Setting \( \bar{y}_6 = 0 \) then gives

\[
\begin{align*}
\bar{y}_5 &= 9, \quad \bar{y}_3 = 16, \quad \bar{y}_1 = 19, \\
\bar{y}_4 &= 12, \quad \bar{y}_2 = 17.
\end{align*}
\]

Check dual feasibility for nonbasic arcs:

- \( \bar{x}_{12} = 3 = u_{12} \), so \( \bar{z}_{12} = 0 \). Then \( \bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 0 \).
- \( \bar{x}_{32} = 0 \), so \( \bar{w}_{32} = 0 \). Then \( \bar{z}_{32} = c_{32} - \bar{y}_3 + \bar{y}_2 = 8 \).
- \( \bar{x}_{34} = 3 = u_{34} \), so \( \bar{z}_{34} = 0 \). Then \( \bar{w}_{34} = -c_{34} + \bar{y}_3 - \bar{y}_4 = -1 \).
- \( \bar{x}_{46} = 5 = u_{46} \), so \( \bar{z}_{46} = 0 \). Then \( \bar{w}_{46} = -c_{46} + \bar{y}_4 - \bar{y}_6 = 5 \).

Thus, **arc (3, 4) enters the basis**, decreasing from its upper bound of 3.
Dual solution

Example 2

B: \( y_i - y_j = c_{ij} \)

NB 0: \( w_{ij} = 0, z_{ij} = c_{ij} - y_i + y_j \geq 0 \)

NB UB: \( z_{ij} = 0, w_{ij} = -c_{ij} + y_i - y_j \geq 0 \)

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Check dual feasibility for nonbasic arcs:

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- \( \bar{x}_{34} = 3 = u_{34}, \) so \( \bar{z}_{34} = 0. \) Then \( \bar{w}_{34} = -c_{34} + \bar{y}_3 - \bar{y}_4 = -1. \)
- \( \bar{x}_{46} = 5 = u_{46}, \) so \( \bar{z}_{46} = 0. \) Then \( \bar{w}_{46} = -c_{46} + \bar{y}_4 - \bar{y}_6 = 5. \)

Thus, \textbf{arc (3, 4) enters the basis}, decreasing from its upper bound of 3.
**Example 2**

Dual solution

Basic arcs give dual values:

\[
\begin{align*}
\bar{x}_{13} \text{ basic implies } & \bar{y}_1 - \bar{y}_3 = 3 \\
\bar{x}_{24} \text{ basic implies } & \bar{y}_2 - \bar{y}_4 = 5 \\
\bar{x}_{35} \text{ basic implies } & \bar{y}_3 - \bar{y}_5 = 7 \\
\bar{x}_{45} \text{ basic implies } & \bar{y}_4 - \bar{y}_5 = 3 \\
\bar{x}_{56} \text{ basic implies } & \bar{y}_5 - \bar{y}_6 = 9 \\
\end{align*}
\]

Setting \( \bar{y}_6 = 0 \) then gives

\[
\begin{align*}
\bar{y}_5 = 9, & \quad \bar{y}_3 = 16, \quad \bar{y}_1 = 19, \\
\bar{y}_4 = 12, & \quad \bar{y}_2 = 17.
\end{align*}
\]

Check dual feasibility for nonbasic arcs:

- \( \bar{x}_{12} = 3 = u_{12}, \) so \( \bar{z}_{12} = 0. \) Then \( \bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 0. \)
- \( \bar{x}_{32} = 0, \) so \( \bar{w}_{32} = 0. \) Then \( \bar{z}_{32} = c_{32} - \bar{y}_3 + \bar{y}_2 = 8. \)
- \( \bar{x}_{34} = 3 = u_{34}, \) so \( \bar{z}_{34} = 0. \) Then \( \bar{w}_{34} = -c_{34} + \bar{y}_3 - \bar{y}_4 = -1. \)
- \( \bar{x}_{46} = 5 = u_{46}, \) so \( \bar{z}_{46} = 0. \) Then \( \bar{w}_{46} = -c_{46} + \bar{y}_4 - \bar{y}_6 = 5. \)

Thus, **arc (3, 4) enters the basis**, decreasing from its upper bound of 3.
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Thus, arc \((3, 4)\) enters the basis, decreasing from its upper bound of 3.
Find cycle, adjust flow
A cycle is created and flow adjusted around the cycle:

triples are \((c_{ij}, u_{ij}, x_{ij})\)

Maximum possible value of \(t\) is 1, with arc \((4, 5)\) leaving the basis.
Updated BFS

The updated BFS is:

triples are \((c_{ij}, u_{ij}, x_{ij})\)

- \((2,3,3)\) UB
- \((3,7,5)\) B
- \((7,4,0)\) LB
- \((7,4,3)\) B
- \((5,6,3)\) B
- \((3,2,0)\) LB
- \((5,3,2)\) B
- \((7,5,5)\) UB
- \((9,8,3)\) B

\(c_{ij}\): cost
\(u_{ij}\): capacity
\(x_{ij}\): flow
Dual solution

Basic arcs give dual values:

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\end{align*}
\]

Setting \( \bar{y}_6 = 0 \) then gives

\[
\begin{align*}
\bar{y}_5 &= 9, \bar{y}_3 = 16, \bar{y}_1 = 19, \\
\bar{y}_4 &= 11, \bar{y}_2 = 16.
\end{align*}
\]

Check dual feasibility for nonbasic arcs:

- \( \bar{x}_{12} = 3 = u_{12} \), so \( \bar{z}_{12} = 0 \). Then \( \bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 1 \).
- \( \bar{x}_{32} = 0 \), so \( \bar{w}_{32} = 0 \). Then \( \bar{z}_{32} = c_{32} - \bar{y}_3 + \bar{y}_2 = 7 \).
- \( \bar{x}_{45} = 0 \), so \( \bar{w}_{45} = 0 \). Then \( \bar{z}_{45} = c_{34} - \bar{y}_3 + \bar{y}_4 = 1 \).
- \( \bar{x}_{46} = 5 = u_{46} \), so \( \bar{z}_{46} = 0 \). Then \( \bar{w}_{46} = -c_{46} + \bar{y}_4 - \bar{y}_6 = 4 \).

Since we are dual feasible, we are now \textbf{optimal}. 

B: \( y_i - y_j = c_{ij} \)

NB 0: \( w_{ij} = 0, z_{ij} = c_{ij} - y_i + y_j \geq 0 \)

NB UB: \( z_{ij} = 0, w_{ij} = -c_{ij} + y_i - y_j \geq 0 \)
Example 2

Dual solution

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Check dual feasibility for nonbasic arcs:

\[ \bar{x}_{12} = 3 = u_{12}, \text{ so } \bar{z}_{12} = 0. \text{ Then } \bar{w}_{12} = -c_{12} + \bar{y}_1 - \bar{y}_2 = 1. \]
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Example 2

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