Math Models of OR: General Network Models

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Outline

1. An example problem
2. The general problem and its dual
3. Solve the example
An example problem

In a transportation problem, we only have two types of nodes: sources and destinations. Further, every edge connects a source to a destination.

In a more general network, we may have transshipment nodes, and an edge can connect any two nodes.

Consider for example the following problem with 5 nodes. The cost of sending one unit of flow along each arc is indicated.

Node 1: Supply 10
Node 2: Supply 10
Node 3: Supply 20
Node 4: Transshipment node
Node 5: Demand 40
**Flow conservation**

The variables are $x_{ij}$ for arcs $(i, j)$. There is a flow conservation constraint at each node, with the form of the constraint depending on the nature of the node.

- **Supply nodes:**
  \[
  \text{flow out} - \text{flow in} = \text{supply}
  \]

- **Demand nodes:**
  \[
  \text{flow out} - \text{flow in} = -\text{demand}
  \]

- **Transshipment nodes:**
  \[
  \text{flow out} - \text{flow in} = 0
  \]

Finding a minimum cost shipment plan for the example problem can then be written as a linear optimization problem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^7} & \quad 2x_{12} + 6x_{13} + x_{24} + 5x_{32} + 3x_{35} + 2x_{43} + 6x_{45} \\
\text{s.t.} & \quad x_{12} + x_{13} = 10 \\
& \quad -x_{12} + x_{24} - x_{32} + x_{35} - x_{43} + x_{43} + x_{45} = 0 \\
& \quad x_{ij} \geq 0 \text{ for each edge } (i,j)
\end{align*}
\]
An example problem

Basic feasible solutions

\[
\begin{align*}
\text{min}_{x \in \mathbb{R}^7} & \quad 2x_{12} + 6x_{13} + x_{24} + 5x_{32} + 3x_{35} + 2x_{43} + 6x_{45} \\
\text{s.t.} & \quad x_{12} + x_{13} = 10 \quad + x_{24} - x_{32} = 10 \quad - x_{13} = 20 \\
& \quad - x_{24} + x_{32} + x_{35} - x_{43} = 0 \\
& \quad - x_{35} + x_{43} + x_{45} = 0 \\
& \quad x_{ij} \geq 0 \text{ for each edge } (i,j)
\end{align*}
\]

Note that each column contains exactly two nonzeros, one “1” and one “-1”. Therefore, adding all the constraints gives the equality \(0 = 0\).

Thus, the constraints are linearly dependent, so one of them could be discarded.
Spanning trees

A basic feasible solution will have basic variables corresponding to a spanning tree:

- the number of basic variables is one less than the number of nodes.
- the set of basic variables does not contain a cycle.
- for any two vertices, there is a path from one vertex to the other using only basic edges (perhaps traversing some edges in the reverse direction).

1

\[ c_{12} = 2, \ x_{12} = 10 \]

2

\[ c_{24} = 1, \ x_{24} = 10 \]

3

\[ c_{13} = 6, \ x_{13} = 10 \]

\[ c_{32} = 5 \]

\[ c_{35} = 3, \ x_{35} = 40 \]

4

\[ c_{43} = 2, \ x_{43} = 10 \]

\[ c_{45} = 6 \]

5

\[ c_{13} = 6, \ x_{13} = 10 \]
Connected graphs

Note: we are assuming here that the original network is *connected*, that is, there is a path between any pair of vertices in the original graph (again, perhaps traversing some edges in the reverse direction).

If it is not connected then it can be broken into components, and a basic feasible solution has basic variables that correspond to spanning trees on each component.

Something not connected
The dual for example problem has one variable for each node and one constraint for each edge and is

\[ \begin{align*}
\max_{y \in \mathbb{R}^5} & \quad 10y_1 + 10y_2 + 20y_3 - 40y_5 \\
\text{subject to} & \quad y_1 - y_2 \leq 2 \\
& \quad -y_1 - y_3 \leq 6 \\
& \quad y_2 - y_4 \leq 1 \\
& \quad -y_2 + y_3 \leq 5 \\
& \quad y_3 - y_5 \leq 3 \\
& \quad -y_3 + y_4 \leq 2 \\
& \quad y_4 - y_5 \leq 6 \\
& \quad y_i \text{ free, } i = 1, \ldots, 5
\end{align*} \]
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The general form of the primal and dual problems

The general network flow problem on a graph $G = (V, E)$ with vertices $V$ and edges $E$ has the form

$$\min_{x \in \mathbb{R}^{|E|}} \sum_{(i,j) \in E} c_{ij} x_{ij}$$

subject to

$$\sum_{j \in V : (i,j) \in E} x_{ij} - \sum_{k \in V : (k,i) \in E} x_{ki} = b_i \quad \text{for all } i \in V$$

$$x_{ij} \geq 0 \quad \text{for all } (i,j) \in E$$

where

$$b_i \begin{cases} > 0 & \text{if } i \text{ is a supply node} \\ < 0 & \text{if } i \text{ is a demand node} \\ = 0 & \text{if } i \text{ is a transshipment node} \end{cases}$$

The dual problem is

$$\max_{y \in \mathbb{R}^{|V|}} \sum_{i \in V} b_i y_i$$

subject to

$$y_i - y_j \leq c_{ij} \quad \text{for all edges } (i,j) \in E$$

$$y \text{ free}$$
Using simplex

We solve the primal problem using simplex, so at each iteration we have a basic feasible solution to the primal problem. We perform the following steps at each iteration:

1. **Construct a dual solution using complementary slackness**, so solve the system of equations $y_i - y_j = c_{ij}$ for all basic variables $x_{ij}$.

2. Calculate the dual slacks $c_{ij} - y_i + y_j$ for all the nonbasic variables $x_{ij}$. These are the reduced costs.

3. If all the reduced costs are nonnegative, STOP, we are optimal.

4. Else, choose a nonbasic variable with a negative reduced cost to enter the basis.

5. The incoming edge creates a unique cycle, since the basic variables constitute a spanning tree. Adjust flow around the cycle to maintain feasibility, until flow drops to 0 on one of the basic edges. This basic edge becomes nonbasic.
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We solve the example using simplex. We initialize with the four basic variables $x_{13} = 10$, $x_{24} = 10$, $x_{35} = 30$, $x_{45} = 10$. 

Initial BFS
Dual variables and reduced costs

Find the dual variables. Note that we have 4 equations in 5 unknowns, so we arbitrarily set $y_5 = 0$.

$$
\begin{align*}
  y_1 - y_3 &= 6 \\
  y_2 - y_4 &= 1 \\
  y_3 - y_5 &= 3 \\
  y_4 - y_5 &= 6
\end{align*}
\implies
\begin{cases}
  \text{Set } y_5 = 0: \\
  \implies y_3 = 3, \ y_4 = 6 \\
  \implies y_1 = 9, \ y_2 = 7
\end{cases}
$$

Reduced costs:

$$
\begin{align*}
  x_{12} : \ c_{12} - y_1 + y_2 &= 2 - 9 + 7 = 0 \checkmark \\
  x_{32} : \ c_{32} - y_3 + y_2 &= 5 - 3 + 7 = 9 \checkmark \\
  x_{43} : \ c_{43} - y_4 + y_3 &= 2 - 6 + 3 = -1 \leftarrow
\end{align*}
$$
Variable $x_{43}$ enters the basis, creating a cycle $4 \rightarrow 3 \rightarrow 5 \rightarrow 4$. 

$$
\begin{align*}
  c_{13} &= 6, \quad x_{13} = 10, \\
  c_{24} &= 1, \quad x_{24} = 10, \\
  c_{35} &= 3, \quad x_{35} = 30 + t, \\
  c_{43} &= 2, \quad x_{43} = t, \\
  c_{45} &= 6, \quad x_{45} = 10 - t
\end{align*}
$$
Update the primal solution

The largest possible flow on edge $x_{43}$ is 10, at which point $x_{45}$ drops to zero. So $x_{45}$ leaves the basis, giving an updated basic feasible solution.
Find the dual solution and reduced costs

Find the dual variables.

\[
\begin{align*}
    y_1 - y_3 &= 6 \\
    y_2 - y_4 &= 1 \\
    y_3 - y_5 &= 3 \\
    y_4 - y_3 &= 2
\end{align*}
\]

Set \( y_5 = 0 \):

\[
\begin{align*}
    y_3 &= 3 \\
    y_1 &= 9, \ y_4 &= 5 \\
    y_2 &= 6
\end{align*}
\]

Reduced costs:

\[
\begin{align*}
    x_{12} : c_{12} - y_1 + y_2 &= 2 - 9 + 6 = -1 \\
    x_{32} : c_{32} - y_3 + y_2 &= 5 - 3 + 6 = 8 \checkmark \\
    x_{45} : c_{45} - y_4 + y_5 &= 6 - 5 + 0 = 1 \checkmark
\end{align*}
\]
Pivot

Variable $x_{12}$ enters the basis, creating a cycle $1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 1$.
Update the primal solution

The largest possible flow on edge $x_{12}$ is 10, at which point $x_{13}$ drops to zero. So $x_{13}$ leaves the basis, giving an updated basic feasible solution.

$$
\begin{align*}
   c_{12} &= 2, \quad x_{12} = 10 \\
   c_{24} &= 1, \quad x_{24} = 20 \\
   c_{32} &= 5 \\
   c_{35} &= 3, \quad x_{35} = 40 \\
   c_{43} &= 2, \quad x_{43} = 20 \\
   c_{45} &= 6
\end{align*}
$$
Find the dual solution and reduced costs

Find the dual variables.

\[
\begin{align*}
    y_1 - y_2 &= 2 \\
    y_2 - y_4 &= 1 \\
    y_3 - y_5 &= 3 \\
    y_4 - y_3 &= 2
\end{align*}
\]

\[
\implies \quad \text{Set } y_5 = 0: \quad \begin{align*}
    y_3 &= 3 \\
    y_4 &= 5 \\
    y_2 &= 6 \\
    y_1 &= 8
\end{align*}
\]

Reduced costs:

\[
\begin{align*}
    x_{13} : \quad &c_{13} - y_1 + y_3 = 6 - 8 + 3 = 1 \checkmark \\
    x_{32} : \quad &c_{32} - y_3 + y_2 = 5 - 3 + 6 = 8 \checkmark \\
    x_{45} : \quad &c_{45} - y_4 + y_5 = 6 - 5 + 0 = 1 \checkmark
\end{align*}
\]

Since all the reduced costs are nonnegative, we are \textbf{optimal}. 
Model with AMPL

Have

set VERTICES;
set EDGES within [VERTICES, VERTICES];

var \( x \in \{\text{EDGES} \} \geq 0 \);

param supply \{\text{VERTICES} \};

subject to flowconservation \{i \in \text{VERTICES} \}:

\[ \sum_{(i,j) \in \text{EDGES}} x_{ij} - \sum_{(k,i) \in \text{EDGES}} x_{ki} = \text{supply}_{i} \]