Math Models of OR: General Network Models

John E. Mitchell

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

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Outline

1. An example problem
2. The general problem and its dual
3. Solve the example
An example problem

In a transportation problem, we only have two types of nodes: sources and destinations. Further, every edge connects a source to a destination.

In a more general network, we may have transshipment nodes, and an edge can connect any two nodes.

Consider for example the following problem with 5 nodes. The cost of sending one unit of flow along each arc is indicated.

Node 1: Supply 10
Node 2: Supply 10
Node 3: Supply 20
Node 4: Transshipment node
Node 5: Demand 40
Flow conservation

The variables are $x_{ij}$ for arcs $(i, j)$. There is a flow conservation constraint at each node, with the form of the constraint depending on the nature of the node.

- flow out $-$ flow in $= \text{supply}$ for supply nodes
- flow out $-$ flow in $= -\text{demand}$ for demand nodes
- flow out $-$ flow in $= 0$ for transshipment nodes

Finding a minimum cost shipment plan for the example problem can then be written as a linear optimization problem:

$$\min_{x \in R^7} \begin{align*}
2x_{12} & + 6x_{13} & + x_{24} & + 5x_{32} & + 3x_{35} & + 2x_{43} & + 6x_{45} \\
\end{align*}$$

s.t. $\begin{align*}
x_{12} & + x_{13} \\
-x_{12} & + x_{24} & - x_{32} \\
-x_{13} & + x_{32} & + x_{35} & - x_{43} \\
-x_{24} & + x_{32} & + x_{35} & + x_{43} & + x_{45} & = 0 \\
-x_{35} & - x_{45} & = -40 \\
\end{align*}$

$x_{ij} \geq 0$ for each edge $(i,j)$
Basic feasible solutions

\[
\begin{align*}
\min_{x \in \mathbb{R}^7} & \quad 2x_{12} + 6x_{13} + x_{24} + 5x_{32} + 3x_{35} + 2x_{43} + 6x_{45} \\
\text{s.t.} & \quad x_{12} + x_{13} \quad + x_{24} - x_{32} - x_{35} - x_{43} + x_{45} = 10 \\
& \quad -x_{12} + x_{13} \quad + x_{24} - x_{32} + x_{35} - x_{43} + x_{45} = 10 \\
& \quad -x_{13} + x_{24} + x_{32} + x_{35} - x_{43} + x_{45} = 20 \\
& \quad -x_{24} - x_{35} - x_{43} - x_{45} = -40 \\
& \quad x_{ij} \geq 0 \quad \text{for each edge} \quad (i,j)
\end{align*}
\]

Note that each column contains exactly two nonzeros, one “1” and one “-1”. Therefore, adding all the constraints gives the equality \(0 = 0\).

Thus, the constraints are linearly dependent, so one of them could be discarded.
Spanning trees
A basic feasible solution will have basic variables corresponding to a spanning tree:

- the number of basic variables is one less than the number of nodes.
- the set of basic variables does not contain a cycle.
- for any two vertices, there is a path from one vertex to the other using only basic edges (perhaps traversing some edges in the reverse direction).

\[
\begin{align*}
\text{1} & \quad c_{12} = 2 & \quad c_{13} = 6, \quad x_{13} = 10 \\
\text{2} & \quad c_{24} = 1, \quad x_{24} = 10 \\
\text{3} & \quad c_{32} = 5 \\
\text{4} & \quad c_{43} = 2, \quad x_{43} = 10 \\
\text{5} & \quad c_{35} = 3, \quad x_{35} = 40 \\
\text{4} & \quad c_{45} = 6 \\
\end{align*}
\]
Connected graphs

Note: we are assuming here that the original network is *connected*, that is, there is a path between any pair of vertices in the original graph (again, perhaps traversing some edges in the reverse direction).

If it is not connected then it can be broken into components, and a basic feasible solution has basic variables that correspond to spanning trees on each component.
Dual problem

The dual for example problem has one variable for each node and one constraint for each edge and is

$$\max_{y \in \mathbb{R}^5} 10y_1 + 10y_2 + 20y_3 - 40y_5$$
subject to

$$y_1 - y_2 \leq 2$$
$$y_1 - y_3 \leq 6$$
$$y_2 - y_4 \leq 1$$
$$-y_2 + y_3 \leq 5$$
$$y_3 - y_5 \leq 3$$
$$-y_3 + y_4 \leq 2$$
$$y_4 - y_5 \leq 6$$

$$y_i \text{ free, } i = 1, \ldots, 5$$
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The general form of the primal and dual problems

The general network flow problem on a graph $G = (V, E)$ with vertices $V$ and edges $E$ has the form

$$\min_{x \in \mathbb{R}^{|E|}} \sum_{(i,j) \in E} c_{ij} x_{ij}$$

subject to

$$\sum_{j \in V : (i,j) \in E} x_{ij} - \sum_{k \in V : (k,i) \in E} x_{ki} = b_i \quad \text{for all } i \in V$$

$$x_{ij} \geq 0 \quad \text{for all } (i,j) \in E$$

where

$$b_i \begin{cases} > 0 & \text{if } i \text{ is a supply node} \\ < 0 & \text{if } i \text{ is a demand node} \\ = 0 & \text{if } i \text{ is a transshipment node} \end{cases}$$

The dual problem is

$$\max_{y \in \mathbb{R}^{|V|}} \sum_{i \in V} b_i y_i$$

subject to

$$y_i - y_j \leq c_{ij} \quad \text{for all edges } (i,j) \in E$$

$y$ free
Using simplex

We solve the primal problem using simplex, so at each iteration we have a basic feasible solution to the primal problem. We perform the following steps at each iteration:

1. **Construct a dual solution using complementary slackness**, so solve the system of equations \( y_i - y_j = c_{ij} \) for all basic variables \( x_{ij} \).

2. Calculate the dual slacks \( c_{ij} - y_i + y_j \) for all the nonbasic variables \( x_{ij} \). These are the reduced costs.

3. If all the reduced costs are nonnegative, STOP, we are optimal.

4. Else, choose a nonbasic variable with a negative reduced cost to enter the basis.

5. The incoming edge creates a unique cycle, since the basic variables constitute a spanning tree. Adjust flow around the cycle to maintain feasibility, until flow drops to 0 on one of the basic edges. This basic edge becomes nonbasic.
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Initial BFS

We solve the example using simplex. We initialize with the four basic variables $x_{13} = 10$, $x_{24} = 10$, $x_{35} = 30$, $x_{45} = 10$. 

### Basic Equations

- Basic: $y_i - y_j = c_{ij}$
- Nonbasic: $y_i - y_j \leq c_{ij}$
Dual variables, reduced costs

Find the dual variables. Note that we have 4 equations in 5 unknowns, so we arbitrarily set $y_5 = 0$.

\[
\begin{align*}
    y_1 - y_3 &= 6 \\
    y_2 - y_4 &= 1 \\
    y_3 - y_5 &= 3 \\
    y_4 - y_5 &= 6
\end{align*}
\]

\[
\begin{align*}
\Rightarrow & \quad \{ \text{Set } y_5 = 0: \\ & \Rightarrow y_3 = 3, \ y_4 = 6 \\
& \Rightarrow y_1 = 9, \ y_2 = 7
\end{align*}
\]

Reduced costs:

\[
\begin{align*}
    x_{12} : \ & c_{12} - y_1 + y_2 = 2 - 9 + 7 = 0 \checkmark \\
    x_{32} : \ & c_{32} - y_3 + y_2 = 5 - 3 + 7 = 9 \checkmark \\
    x_{43} : \ & c_{43} - y_4 + y_3 = 2 - 6 + 3 = -1 \leftarrow
\end{align*}
\]
Solve the example

Pivot

Variable $x_{43}$ enters the basis, creating a cycle $4 \rightarrow 3 \rightarrow 5 \rightarrow 4$. 

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**Basic:** $y_i - y_j = c_{ij}$

**Nonbasic:** $y_i - y_j \leq c_{ij}$

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Graphical representation of the network with variable values and constraints.

1. $c_{12} = 2$, $x_{12} = t$
2. $c_{13} = 6$, $x_{13} = 10$
3. $c_{24} = 1$, $x_{24} = 10$
4. $c_{35} = 3$, $x_{35} = 30 + t$
5. $c_{43} = 2$, $x_{43} = t$
6. $c_{45} = 6$, $x_{45} = 10 - t$
Update the primal solution

The largest possible flow on edge $x_{43}$ is 10, at which point $x_{45}$ drops to zero. So $x_{45}$ leaves the basis, giving an updated basic feasible solution.
Solve the example

Find dual solution, reduced costs

Find the dual variables.

\begin{align*}
    y_1 - y_3 &= 6 \\
    y_2 - y_4 &= 1 \\
    y_3 - y_5 &= 3 \\
    y_4 - y_3 &= 2
\end{align*}

\begin{itemize}
    \item Set \( y_5 = 0 \):
    \begin{align*}
        \implies y_3 &= 3 \\
        \implies y_1 &= 9, \ y_4 = 5 \\
        \implies y_2 &= 6
    \end{align*}
\end{itemize}

Reduced costs:

\begin{align*}
    x_{12} : & \quad c_{12} - y_1 + y_2 = 2 - 9 + 6 = -1 \leftarrow \\
    x_{32} : & \quad c_{32} - y_3 + y_2 = 5 - 3 + 6 = 8 \checkmark \\
    x_{45} : & \quad c_{45} - y_4 + y_5 = 6 - 5 + 0 = 1 \checkmark
\end{align*}
Solve the example

**Pivot**

basic: \( y_i - y_j = c_{ij} \)
nonbasic: \( y_i - y_j \leq c_{ij} \)

Variable \( x_{12} \) enters the basis, creating a cycle 1 – 2 – 4 – 3 – 1.

1. \( c_{13} = 6, x_{13} = 10 - t \)
2. \( c_{12} = 2, x_{12} = t \)
3. \( c_{24} = 1, x_{24} = 10 + t \)
4. \( c_{32} = 5 \)
5. \( c_{35} = 3, x_{35} = 40 \)
6. \( c_{43} = 2, x_{43} = 10 + t \)
7. \( c_{45} = 6 \)
Update the primal solution

The largest possible flow on edge $x_{12}$ is 10, at which point $x_{13}$ drops to zero. So $x_{13}$ leaves the basis, giving an updated basic feasible solution.

$$c_{12} = 2, \quad x_{12} = 10$$
$$c_{24} = 1, \quad x_{24} = 20$$
$$c_{32} = 5$$
$$c_{35} = 3, \quad x_{35} = 40$$
$$c_{43} = 2, \quad x_{43} = 20$$
$$c_{45} = 6$$
$$c_{13} = 6$$
Find dual solution, reduced costs

Find the dual variables.

\[
\begin{align*}
y_1 - y_2 &= 2 \\
y_2 - y_4 &= 1 \\
y_3 - y_5 &= 3 \\
y_4 - y_3 &= 2
\end{align*}
\]

Set \( y_5 = 0 \):

\[
\begin{align*}
\implies y_3 &= 3 \\
\implies y_4 &= 5 \\
\implies y_2 &= 6 \\
\implies y_1 &= 8
\end{align*}
\]

Reduced costs:

\[
\begin{align*}
x_{13} : \ c_{13} - y_1 + y_3 &= 6 - 8 + 3 = 1 \\
x_{32} : \ c_{32} - y_3 + y_2 &= 5 - 3 + 6 = 8 \\
x_{45} : \ c_{45} - y_4 + y_5 &= 6 - 5 + 0 = 1
\end{align*}
\]

Since all the reduced costs are nonnegative, we are optimal.