Math Models of OR:
DSES DQE Question on Deterministic OR

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A linear optimization problem

Consider the linear programming problem

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 + x_3 + 4x_4 + 3x_5 \\
\text{s.t.} & \quad -x_1 + x_3 + 3x_4 - x_5 = b_1 \\
& \quad x_1 + x_2 - x_5 = b_2 \\
& \quad 2x_2 + x_3 + 2x_4 + x_5 = b_1 + 2b_2 \\
& \quad x_i \geq 0, \quad i = 1, \ldots, 5.
\end{align*}
\]

(P)

In what follows, assume that the problem \( (P) \) has an optimal solution \( \bar{x} \), where \( \bar{x}_1, \bar{x}_2 \) and \( \bar{x}_3 \) are basic. You do not need to find \( \bar{x} \).
The questions

1. What is the dual to this linear program?
2. Use complementary slackness to find an optimal solution \( \bar{y} \) to the dual problem.
3. Looking purely at the optimal dual solution you found above, can you conclude that no optimal primal solution will have \( x_4 > 0 \)? What about \( x_5 \)?
4. The optimal dual solution \( \bar{y} \) is not unique: there are other optimal solutions of the form \( \bar{y} + \lambda[-1, -2, 1]^T \). What is the largest possible value of \( \lambda \)? Do these other optimal solutions imply anything about the values of \( \bar{x}_1, \bar{x}_2 \) and \( \bar{x}_3 \)? Do they imply anything about the values of \( x_4 \) and \( x_5 \) in any optimal solution to \((P)\)?
The questions

1. What is the dual to this linear program?

2. Use complementary slackness to find an optimal solution $\bar{y}$ to the dual problem.

3. Looking purely at the optimal dual solution you found above, can you conclude that no optimal primal solution will have $x_4 > 0$? What about $x_5$?

4. The optimal dual solution $\bar{y}$ is not unique: there are other optimal solutions of the form $\bar{y} + \lambda [-1, -2, 1]^T$. What is the largest possible value of $\lambda$? Do these other optimal solutions imply anything about the values of $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$? Do they imply anything about the values of $x_4$ and $x_5$ in any optimal solution to $(P)$?
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What is the dual?

\[ y = (2, 3, -1) \]

\[
\begin{align*}
\text{max} & \quad b_1 y_1 + b_2 y_2 + (b_1 + 2b_2) y_3 \\
\text{subject to} & \quad -y_1 + y_2 \leq 1 \\
& \quad y_2 + 2y_3 \leq 1 \\
& \quad y_1 + y_3 \leq 1 \\
& \quad 3y_1 + 2y_3 \leq 4 \\
& \quad -y_1 - y_2 \leq 3 \\
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad c^x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad b^y \\
\text{s.t.} & \quad A^y \leq c
\end{align*}
\]
Find an optimal dual solution

Since there is an optimal basic feasible solution (bfs) with $x_1$, $x_2$ and $x_3$ basic, there must be an optimal dual solution where the first three constraints hold at equality.

Thus, we find an optimal dual solution by solving the equations

\[-y_1 + y_2 = 1\]
\[y_2 + 2y_3 = 1\]
\[y_1 + y_3 = 1\]

Solving these equations gives $\bar{y}_1 = 2$, $\bar{y}_2 = 3$, $\bar{y}_3 = -1$. It can be verified that this solution is dual feasible.
Implications for optimal primal solutions

Every primal optimal solution is complementary to every dual optimal solution.

Thus, any primal optimal solution must be complementary to $\bar{y}$.

The dual constraint corresponding to $x_4$ is

$$3y_1 + 2y_3 \leq 4$$

This constraint is satisfied at equality by the dual solution $\bar{y} = (2, 3, -1)$ found above. Therefore, primal solutions with $x_4 > 0$ may satisfy complementary slackness with respect to $\bar{y}$ and so such solutions may be primal optimal.

However, the dual solution $\bar{y}$ satisfies the last constraint $-y_1 - y_2 + y_3 \leq 3$ strictly; the dual slack is 9.

Thus, no primal optimal solution can have $x_5 > 0$. 
Exploiting other optimal dual solutions

Any primal optimal solution is complementary to any dual optimal solution. If we let $c$ denote the primal objective function, $A$ denote the primal constraint matrix, and $w$ denote the dual slacks, then along the given dual set of optimal solutions we have

$$w = c - A^T y = c - A^T \bar{y} - \lambda A^T \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9 \end{bmatrix} - \lambda \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + \lambda \\ 0 + 0 \\ 0 + 0 \\ 9 - 4\lambda \end{bmatrix}$$

$\lambda \leq \frac{9}{4}$ to keep $w \geq 0$
Using the dual slacks and complementary slackness

The dual slacks for the given optimal set of optimal dual solutions are

\[
\mathbf{w} = \begin{bmatrix}
0 + \lambda \\
0 + 0 \\
0 + 0 \\
0 + \lambda \\
9 - 4\lambda
\end{bmatrix}
\]

This is dual feasible for \( 0 \leq \lambda \leq 9/4 \).

Note that \( w_1 > 0, w_4 > 0, w_5 > 0 \) for \( 0 < \lambda < 9/4 \).

Thus, by complementary slackness, we must have \( x_1 = x_4 = x_5 = 0 \) in any primal optimal solution.

Therefore, \( \bar{x}_1 = 0 \). We can not conclude anything about \( \bar{x}_2 \) or \( \bar{x}_3 \).

This problem has an optimal solution that is primal degenerate, with multiple dual optimal solutions.