Math Models of OR: Complementary Slackness

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Outline

1. A standard dual pair example
2. Another example
3. The general case
4. The simplex algorithm
Consider the standard dual pair example:

\[
\begin{align*}
\min_x & \quad 2x_1 + 5x_2 + 4x_3 \\
\text{s.t.} & \quad x_1 + x_2 - 3x_3 \geq 3 \\
& \quad -x_1 + x_2 + x_3 \geq 1 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_y & \quad 3y_1 + y_2 \\
\text{s.t.} & \quad y_1 - y_2 \leq 2 \\
& \quad y_1 + y_2 \leq 5 \\
& \quad -3y_1 + y_2 \leq 4 \\
& \quad y_1, y_2 \geq 0
\end{align*}
\]
Minimization problem

Initial tableau:

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(s_1)</th>
<th>(s_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>(-3)</td>
<td>(-1)</td>
</tr>
<tr>
<td>1</td>
<td>(-1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Final tableau:

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(s_1)</th>
<th>(s_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-12)</td>
<td>0</td>
<td>0</td>
<td>13</td>
<td>(\frac{7}{2})</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>(-1)</td>
<td>(-\frac{1}{2})</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>(-2)</td>
<td>(-\frac{1}{2})</td>
</tr>
</tbody>
</table>
Maximization problem

Make it into a minimization problem so we can set up a simplex tableau, by changing the sign of the objective:

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
### Relationship between optimal variables and slacks

<table>
<thead>
<tr>
<th>index</th>
<th>primal variable</th>
<th>dual slack</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(x_1^* = 1)</td>
<td>(w_1^* = 0)</td>
</tr>
<tr>
<td>2</td>
<td>(x_2^* = 2)</td>
<td>(w_2^* = 0)</td>
</tr>
<tr>
<td>3</td>
<td>(x_3^* = 0)</td>
<td>(w_3^* = 13)</td>
</tr>
</tbody>
</table>

For each component \(j\), either \(x_j^* = 0\) or \(w_j^* = 0\).

<table>
<thead>
<tr>
<th>index</th>
<th>dual variable</th>
<th>primal slack</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(y_1^* = \frac{7}{2})</td>
<td>(s_1^* = 0)</td>
</tr>
<tr>
<td>2</td>
<td>(y_2^* = \frac{3}{2})</td>
<td>(s_2^* = 0)</td>
</tr>
</tbody>
</table>

For each component \(i = 1, 2\), either \(y_i^* = 0\) or \(s_i^* = 0\).

This relationship always holds and is known as **complementary slackness**.

Note that the “or” is not an “exclusive or”: it is possible for \(x_j^* = w_j^* = 0\) for some component \(j\), and/or for \(y_i = s_i = 0\) for some component \(i\).
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A problem with multiple optimal solutions

\[
\begin{align*}
\text{min}_{x} & \quad x_1 + x_2 + x_3 \\
\text{subject to} & \quad 2x_1 + 3x_2 + x_3 \geq 4 \\
& \quad 2x_1 + x_2 + 3x_3 \geq 4 \\
& \quad x_1 + x_2 + x_3 \geq 2 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 3
\end{align*}
\]
Dual problem

\[
\begin{align*}
\min_{y,s} & \quad 4y_1 + 4y_2 + 2y_3 \\
\text{subject to} & \quad 2y_1 + 2y_2 + y_3 \leq 1 \\
& \quad 3y_1 + y_2 + y_3 \leq 1 \\
& \quad y_1 + 3y_2 + y_3 \leq 1 \\
& \quad y_i \geq 0, \quad i = 1, \ldots, 3
\end{align*}
\]
One pair of optimal solutions

Primal optimal solutions include \( x^* = (2, 0, 0) \), with slacks \( s^* = (0, 0, 0) \).
Dual optimal solutions include \( y^* = (0, 0, 1) \), with slacks \( w^* = (0, 0, 0) \).

These points satisfy complementary slackness:

<table>
<thead>
<tr>
<th>index</th>
<th>primal variable</th>
<th>dual slack</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_1^* = 2 )</td>
<td>( w_1^* = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( x_2^* = 0 )</td>
<td>( w_2^* = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( x_3^* = 0 )</td>
<td>( w_3^* = 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>index</th>
<th>dual variable</th>
<th>primal slack</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( y_1^* = 0 )</td>
<td>( s_1^* = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( y_2^* = 0 )</td>
<td>( s_2^* = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( y_3^* = 1 )</td>
<td>( s_3^* = 0 )</td>
</tr>
</tbody>
</table>
Another pair of optimal solutions

The primal solution \( x^* = (1, 0.5, 0.5) \) with slacks \( s^* = (0, 0, 0) \), and
dual solution \( y^* = (0.125, 0.125, 0.5) \) with slacks \( w^* = (0, 0, 0) \).

Again, this pair satisfies complementary slackness.

This pair satisfies *strict complementarity*: for each index, exactly one of
\( x_j^* \) and \( w_j^* \) is positive, and exactly one of \( y_i^* \) and \( s_i^* \) is positive:

<table>
<thead>
<tr>
<th>index</th>
<th>primal variable</th>
<th>dual slack</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_1^* = 1 )</td>
<td>( w_1^* = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( x_2^* = 0.5 )</td>
<td>( w_2^* = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( x_3^* = 0.5 )</td>
<td>( w_3^* = 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>index</th>
<th>dual variable</th>
<th>primal slack</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( y_1^* = 0.125 )</td>
<td>( s_1^* = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( y_2^* = 0.125 )</td>
<td>( s_2^* = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( y_3^* = 0.5 )</td>
<td>( s_3^* = 0 )</td>
</tr>
</tbody>
</table>
Strict complementarity

If a linear optimization problem has an optimal solution then it has an optimal solution that satisfies strict complementarity.

Every primal optimal solution satisfies complementary slackness with every dual optimal solution.
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The general case

We work with the standard form primal-dual pair of linear optimization problems:

$$\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*} \quad \text{(P)}$$

$$\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \\
& \quad y \geq 0
\end{align*} \quad \text{(D)}$$

Complementary slackness

**Theorem**

A pair of primal and dual feasible solutions are optimal to their respective problems in a primal-dual pair of LPs if and only if

whenever these variables make a slack variable in one problem strictly positive, the value of the associated nonnegative variable in the other is zero.

\{ complementary slackness \}
Proof of theorem

We can define the vector of primal slacks \( s = Ax - b \) for any \( x \in \mathbb{R}^n \) and the vector of dual slacks \( w = c - A^T y \in \mathbb{R}^n \) for any \( y \in \mathbb{R}^m \). Note that the duality gap is

\[
    c^T x - b^T y = c^T x - (Ax - s)^T y = s^T y + c^T x - x^T A^T y \\
    = s^T y + c^T x - (A^T y)^T x = s^T y + (c - A^T y)^T x \\
    = s^T y + w^T x \\
    = \sum_{i=1}^{m} s_i y_i + \sum_{j=1}^{n} w_j x_j
\]

for any \( y \in \mathbb{R}^m \) and \( x \in \mathbb{R}^n \).
Proof (continued)

Thus we have the duality gap:

\[ c^T x - b^T y = s^T y + w^T x = \sum_{i=1}^{m} s_i y_i + \sum_{j=1}^{n} w_j x_j. \]

Note that if \( x \) and \( y \) are feasible in their respective problems then \( x \geq 0, y \geq 0, w \geq 0, \) and \( s \geq 0, \) so \( w^T x \geq 0 \) and \( s^T y \geq 0. \)

If the points are optimal then \( c^T x - b^T y = 0 \) so \( \sum_{i=1}^{m} y_i s_i = 0 \) and \( \sum_{j=1}^{n} w_j x_j = 0, \) so each component \( w_j x_j = 0 \) and each component \( s_i y_i = 0, \) since they must all be nonnegative.

So either \( w_j = 0 \) or \( x_j = 0 \) for each component \( j = 1, \ldots, n, \) and either \( s_i = 0 \) or \( y_i = 0 \) for each component \( i = 1, \ldots, m. \)

This is complementary slackness.
Proof (part 3)

We need to prove the converse. We again exploit the equality

\[ c^T x - b^T y = s^T y + w^T x = \sum_{i=1}^{m} s_i y_i + \sum_{j=1}^{n} w_j x_j. \]

If complementary slackness holds then either \( x_j = 0 \) or \( w_j = 0 \) for each component \( j = 1, \ldots, n \), so \( w^T x = 0 \).

Further, either \( y_i = 0 \) or \( s_i = 0 \) for each component \( i = 1, \ldots, m \), so \( y^T s = 0 \).

It follows that \( c^T x = b^T y \) so the points are optimal.
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Optimality and complementary slackness

The simplex algorithm is stated in terms of the linear optimization problem

\[
\min_{x \in \mathbb{R}^n} \quad c^T x \\
\text{subject to} \quad Ax = b \quad (\hat{P}) \quad \text{with dual} \quad \max_{y \in \mathbb{R}^m} \quad b^T y \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x \geq 0 \\
\]

A pair of points \( \hat{x} \in \mathbb{R}^n \) and \( \hat{y} \in \mathbb{R}^m \) is \textbf{optimal} if and only if it satisfies the following three conditions:

- **Primal feasibility**: \( A\hat{x} = b, \hat{x} \geq 0 \).
- **Dual feasibility**: \( A^T\hat{y} \leq c \).
- **Complementary slackness**: Let \( \hat{w} = c - A^T\hat{y} \). Then \( \hat{x}_j\hat{w}_j = 0 \) for \( j = 1, \ldots, n \).
The simplex algorithm is stated in terms of the linear optimization problem

\[
\min_{x \in \mathbb{R}^n} \quad c^T x \\
\text{subject to} \quad Ax = b \quad (\hat{P}) \quad \text{with dual} \quad \max_{y \in \mathbb{R}^m} \quad b^T y \\
\quad \quad \quad \quad x \geq 0
\]

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- **Complementary slackness**: Let \( \hat{w} = c - A^T\hat{y} \). Then \( \hat{x}_j \hat{w}_j = 0 \) for \( j = 1, \ldots, n \).
Optimality and complementary slackness

The simplex algorithm is stated in terms of the linear optimization problem

$$\begin{align*}
\min_{x \in \mathbb{IR}^n} & \quad c^T x & \quad \max_{y \in \mathbb{IR}^m} & \quad b^T y \\
\text{subject to} & \quad Ax = b & \text{with dual subject to} & \quad A^T y \leq c \\
& \quad x \geq 0 & & \\
\end{align*}$$

A pair of points $\hat{x} \in \mathbb{IR}^n$ and $\hat{y} \in \mathbb{IR}^m$ is optimal if and only if it satisfies the following three conditions:

- **Primal feasibility**: $A\hat{x} = b$, $\hat{x} \geq 0$.
- **Dual feasibility**: $A^T\hat{y} \leq c$.
- **Complementary slackness**: Let $\hat{w} = c - A^T\hat{y}$. Then $\hat{x}_j\hat{w}_j = 0$ for $j = 1, \ldots, n$. 
Simplex and complementary slackness

Once Phase I is complete, the simplex algorithm always *maintains primal feasibility*.

Further, at each iteration, we can construct a dual solution that *satisfies complementary slackness*:

\[ \text{require the dual slack } \hat{w}_j = 0 \text{ for each basic variable } \hat{x}_j. \]

This requires solving a system of \( m \) equations in the \( m \) components of \( \hat{y} \), one equation for each basic variable.

Thus, we’re only lacking dual feasibility. Now, the reduced costs are equal to the dual slacks, when \( \hat{y} \) is constructed in this way. So, iterating to get the dual slacks nonnegative is equivalent to *working towards dual feasibility*.