Math Models of OR: Shadow Prices

John E. Mitchell
http://www.rpi.edu/~mitchj

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

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An economic interpretation of dual variables

A company makes four products from three resources. It looks to solve the following linear optimization problem:

\[
\begin{aligned}
\text{max } & \quad 6x_1 + x_2 + 4x_3 + 5x_4 \\
\text{subject to } & \quad 2x_1 + x_2 + x_3 + x_4 \leq 30 \quad \text{Resource A} \\
& \quad x_1 + 2x_3 + x_4 \leq 15 \quad \text{Resource B} \\
& \quad x_1 + x_2 + x_3 \leq 5 \quad \text{Resource C} \\
& \quad x_j \geq 0 \quad j = 1, \ldots, 4
\end{aligned}
\]  

(1)
Turning into standard form

Turning the problem into a minimization problem by negating the objective function, we can get a canonical form after the addition of slack variables:

\[
M = \begin{bmatrix}
0 & -6 & -1 & -4 & -5 & 0 & 0 & 0 \\
30 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
15 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\
5 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Suppose that prior to production, someone offered to buy one unit of Resource B. At what price would the company be willing to make this sale? Need to tradeoff the gain in revenue from the sale of the resource against the loss of revenue from the impact on production of the four products.

What is the minimum price the company could charge for one unit of Resource B and still keep total revenue of at least $80? Note that $s_2^* = 0$, so current production schedule uses all of Resource B.
Writing the tableau as a linear optimization problem

\[ M^* = \begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90
\end{array} \]

The optimal tableau corresponds to the linear optimization problem

\[
\begin{align*}
\text{max}_{x \in \mathbb{R}^4, s \in \mathbb{R}^3} & \quad -7x_3 - 5s_2 - s_3 + 80 \\
\text{subject to} & \quad -2x_3 + s_1 - s_2 - s_3 = 10 \\
& \quad -x_2 + x_3 + x_4 + s_2 - s_3 = 10 \\
& \quad x_1 + x_2 + x_3 + s_3 = 5 \\
& \quad x_j \geq 0, j = 1, \ldots, 5, \quad s_i \geq 0, i = 1, 2, 3
\end{align*}
\]

Any feasible solution to the original LOP is feasible in this problem with the same objective function value, and vice versa.
Think of $s_2$ as fixed

$max_{x \in \mathbb{R}^4, s \in \mathbb{R}^3} \quad -7x_3 - 5s_2 - s_3 + 80$

subject to

$-2x_3 + s_1 - s_2 - s_3 = 10$

$-x_2 + x_3 + x_4 + s_2 - s_3 = 10$

$x_1 + x_2 + x_3 + s_3 = 5$

$x_j \geq 0, \ j = 1, \ldots, 5, \ s_i \geq 0, \ i = 1, 2, 3$

We could think of $s_2$ taking a specified value, and then ensuring the other variables are adjusted to maintain feasibility. We could write this by putting $s_2$ on the right hand side:

$max_{x \in \mathbb{R}^4, s \in \mathbb{R}^3} \quad -7x_3 - s_3 + (80 - 5s_2)$

subject to

$-2x_3 + s_1 - s_3 = 10 + s_2$

$-x_2 + x_3 + x_4 - s_3 = 10 - s_2$

$x_1 + x_2 + x_3 + s_3 = 5$

$x_j \geq 0, \ j = 1, \ldots, 5, \ s_i \geq 0, \ i = 1, 3$
A simplex tableau with $s_2$ fixed

$$\begin{align*}
\max_{x \in \mathbb{R}^4, s \in \mathbb{R}^3} & \quad -7x_3 - s_3 + (80 - 5s_2) \\
\text{subject to} & \quad -2x_3 + s_1 - s_3 = 10 + s_2 \\
& \quad -x_2 + x_3 + x_4 - s_3 = 10 - s_2 \\
& \quad x_1 + x_2 + x_3 + s_3 = 5 \\
& \quad x_j \geq 0, \; j = 1, \ldots, 5, \quad s_i \geq 0, \; i = 1, 3
\end{align*}$$

Keeping $s_2$ fixed, we have a simplex tableau for this problem:

$$M' = \begin{pmatrix}
80 - 5s_2 & 0 & 0 & 7 & 0 & 0 & 1 \\
10 + s_2 & 0 & 0 & -2 & 0 & 1 & -1 \\
10 - s_2 & 0 & -1 & 1 & 1 & 0 & -1 \\
5 & 1 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}$$

This is in optimal form provided $-10 \leq s_2 \leq 10$. 
Analyzing the tableau

\[
M' = \begin{bmatrix}
80 - 5s_2 & 0 & 0 & 7 & 0 & 0 & 1 \\
10 + s_2 & 0 & 0 & -2 & 0 & 1 & -1 \\
10 - s_2 & 0 & -1 & 1 & 0 & 0 & -1 \\
5 & 1 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

In terms of the original maximization problem, the objective function value is \( (80 - 5s_2) \); this is the revenue from the production schedule given that we need to keep \( s_2 \) units of Resource B in reserve.

We have:

\[
\text{total revenue} = (\text{revenue from production}) + (\text{revenue from sale of Resource B}) \\
= (80 - 5s_2) + (\text{selling price per unit of Resource B}) \times s_2
\]

Thus, we need the selling price per unit of Resource B to be at least $5 to make it attractive to sell the resource.
Shadow prices

We need the selling price per unit of Resource B to be at least $5 to make it attractive to sell the resource.

This is the shadow price or imputed price for the resource. It is the reduced cost of the slack variable.
The dual problem

The dual problem is

$$\min_{y \in \mathbb{R}^3} \quad 30y_1 + 15y_2 + 5y_3$$

subject to

$$\begin{align*}
2y_1 + y_2 + y_3 & \geq 6 \\
y_1 + y_3 & \geq 1 \\
y_1 + 2y_2 + y_3 & \geq 4 \\
y_1 + y_2 & \geq 5 \\
y_1, y_2, y_3 & \geq 0
\end{align*}$$

which has optimal solution $y^* = (0, 5, 1)$.

Optimal primal tableau:

$$M^* = \begin{bmatrix}
\begin{array}{ccccccccc}
80 & 0 & 0 & 7 & 0 & 0 & 5 & 1 \\
10 & 0 & 0 & -2 & 0 & 1 & -1 & -1 \\
10 & 0 & -1 & 1 & 1 & 0 & 1 & -1 \\
5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\end{bmatrix}$$
Shadow prices and dual variables

Optimal dual solution: \( y^* = (0, 5, 1) \).

Optimal reduced costs for primal slack variables:

\[
\begin{align*}
s_1 & : 0, \\
s_2 & : 5, \\
s_3 & : 1
\end{align*}
\]

Notice for a production problem of this form, the dual variables are exactly equal to the reduced costs for the primal slack variables.

We extend the terminology “shadow prices” to other linear optimization problems to refer to the dual variables.

In the example, the shadow price for Resource C is $1. The slack variable for Resource A is positive at optimality, so not all of this resource is needed, so its shadow price is 0.