Math Models of OR: Duality Relations

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Standard primal-dual pair

Our standard **primal-dual pair** of linear optimization problems consists of a primal problem (P) that is a minimization problem and a dual problem (D) that is a maximization problem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \quad (P) \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \\
& \quad y \geq 0
\end{align*}
\]

Notice how the roles of \( b \) and \( c \) have been reversed, and how \( A \) has been replaced by its transpose. Let \( A \) be an \( m \times n \) matrix. The dimension of the variables in the primal problem (P) is equal to \( n \), the number of **columns** of \( A \). The dimension of the variables in the dual problem (D) is equal to \( m \), the number of **rows** of \( A \).

We give some properties of the primal-dual pair (P) and (D) below. Every LP has a dual problem, and similar properties can be derived for any primal-dual pair.
Weak duality

If $x$ is feasible in (P) and $y$ is feasible in (D) then $c^T x \geq b^T y$. This holds since:

We have $c \geq A^T y$ and $x \geq 0 \implies c^T x \geq x^T A^T y$.

Writing component by component, we have $c_i \geq (A^T y)_i$ and $x_i \geq 0$, so $c_i x_i \geq x_i (A^T y)_i \forall i$, so $c^T x = \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} x_i (A^T y)_i = x^T A^T y$.

Also, $Ax \geq b$ and $y \geq 0 \implies y^T Ax \geq b^T y$.

Note that $x^T A^T y$ is a scalar (so, a $1 \times 1$ matrix), so it’s equal to its transpose.

So, $x^T A^T y = y^T Ax$, so $c^T x \geq x^T A^T y = y^T Ax \geq b^T y$. 
Certificate of optimality

If $\bar{x}$ is feasible in (P) and $\bar{y}$ is feasible in (D) and $c^T \bar{x} = b^T \bar{y}$ then $\bar{x}$ solves (P) and $\bar{y}$ solves (D).

Since: we know $c^T x \geq b^T \bar{y}$ for any $x$ feasible in (P).
So $c^T x \geq c^T \bar{x}$ for any feasible $x$.

Similarly, any $y$ feasible in (D) satisfies $b^T y \leq c^T \bar{x} = b^T \bar{y}$. 
Strong duality

If one of the problems has an optimal solution then the other one also has an optimal solution, and **their optimal values are the same**.

Next time: we will see how to construct an optimal dual solution from an optimal primal solution. (More precisely, from an optimal primal solution that is a BFS to a standard form version of \(P\).)
Achieving optimal values

If both problems are feasible then both problems have optimal solutions, and their optimal values agree.

*Note: consider the nonlinear optimization problem*

$$\min_{x \in \mathbb{R}} e^{-x}.$$

*This problem is feasible: take any $x \in \mathbb{R}$. Every feasible solution has positive objective function value, so the optimal value is bounded. However, the optimal value of 0 is not attained: every $x$ has $e^{-x} > 0$.*

So the following claim is not true:

**Claim:** (does not hold) If a minimization problem is feasible and its objective value is bounded below then it has an optimal solution.

This claim is true for linear optimization problems.
Unboundedness and infeasibility

If (P) has an unbounded optimal value then (D) is infeasible.

Since: We can drive \( c^T x \to -\infty \), and any feasible \( y \) for (D) would have to have \( b^T y \leq c^T x \). So \( b^T y \) cannot take a finite value.

Similarly, if (D) has an unbounded optimal value then (P) is infeasible.

It is possible for both (P) and (D) to be infeasible. We have an example with \( m = n = 1 \), where we take \( A \) to be the \( 1 \times 1 \) matrix with entry 0:

\[
\begin{align*}
\min_{x \in \mathbb{R}^1} & \quad -x \\
\text{subject to} & \quad 0x \geq 1 \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y \in \mathbb{R}^1} & \quad b^T y \\
\text{subject to} & \quad 0y \leq -1 \\
& \quad y \geq 0.
\end{align*}
\]
Complementary slackness

Each dual constraint corresponds to a primal variable.

Next time, we’ll see that either $x_i = 0$ or the corresponding dual slack is zero in any optimal pair to (P) and (D).

Similarly, each primal constraint corresponds to a dual variable, and either $y_j = 0$ or the corresponding primal slack must be zero at any optimal pair to (P) and (D).