Math Models of OR: Motivation for Duality

John E. Mitchell

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

October 2018
Outline

1. The Diet Problem

2. Combinations of constraints
The diet problem

Find the cheapest combination of foods that will satisfy our **daily nutritional requirements**.

For example, want to eat at least 90g of protein per day, want to consume no more than 2000mg of sodium per day.

An 8oz glass of 2% milk contains 8.1g of protein and 100mg of sodium, and costs $0.40.
Parameters

For each food $j = 1, \ldots, n$, need to know:

- Cost $c_j$ per unit serving
- Nutritional composition per unit serving: let $a_{ij}$ denote the amount of nutrient $i$ in one serving of food $j$, for $i = 1, \ldots, m$.

We have daily requirements for the $m$ nutrients: protein, carbohydrates, vitamin A, etc.

To make things simple, we’ll assume all these requirements are lower bounds: we have to consume at least a certain amount $b_i$ of nutrient $i$ for $i = 1, \ldots, m$. 
Parameters

For each food $j = 1, \ldots, n$, need to know:

- Cost $c_j$ per unit serving
- Nutritional composition per unit serving: let $a_{ij}$ denote the amount of nutrient $i$ in one serving of food $j$, for $i = 1, \ldots, m$.

We have daily requirements for the $m$ nutrients: protein, carbohydrates, vitamin A, etc.

To make things simple, we’ll assume all these requirements are lower bounds: we have to consume at least a certain amount $b_i$ of nutrient $i$ for $i = 1, \ldots, m$. 
Objective and constraints

Decision variables: $x_j$: daily consumption of food $j$.

Total daily cost: $\sum_j c_j x_j$

Constraints:

- *Meet nutritional requirements*: Total daily consumption of nutrient $i$ is $\sum_j a_{ij} x_j$. So get constraints

  $$\sum_j a_{ij} x_j \geq b_i \quad \text{for each nutrient } i.$$

- *Nonnegativity*: $x_j \geq 0$ for each food $j$. 

Objective and constraints

Decision variables: \( x_j \): daily consumption of food \( j \).

Total daily cost: \( \sum_j c_j x_j \)

Constraints:

- *Meet nutritional requirements*: Total daily consumption of nutrient \( i \) is \( \sum_j a_{ij} x_j \). So get constraints

\[
\sum_j a_{ij} x_j \geq b_i \quad \text{for each nutrient } i.
\]

- *Nonnegativity*: \( x_j \geq 0 \) for each food \( j \).
Formulation as a linear optimization problem:

To simplify, we assume the foodstuffs are infinitely divisible, so we can use continuous variables. One unit of foodstuff $j$ costs $c_j$ and provides $a_{ij}$ units of nutrient $i$.

$$\min_{x \in \mathbb{R}^n} \sum_{j=1}^{n} c_j x_j$$
$$\text{subject to} \quad \sum_j a_{ij} x_j \geq b_i \quad \text{for each nutrient } i = 1, \ldots, m$$
$$x_j \geq 0 \quad \text{for each food } j = 1, \ldots, n$$

or equivalently

$$\min_x c^T x$$
$$\text{subject to} \quad Ax \geq b$$
$$x \geq 0.$$
A dual problem

Now consider a manufacturer of nutrient powders, for \( i = 1, \ldots, m \).

How should a manufacturer choose prices \( y_i \) for the powders so the consumer buys the powders instead of the foodstuffs? (Assume the consumer only cares about cost and is indifferent to taste.)

To replace foodstuff \( j \), need

\[
\sum_{i=1}^{m} a_{ij} y_i \leq c_j,
\]

since one unit of foodstuff \( j \) costs \( c_j \) and provides \( a_{ij} \) of nutrient \( i \).

<table>
<thead>
<tr>
<th>one unit of foodstuff ( j )</th>
<th>amount of nutrient ( i )</th>
<th>cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{ij} ) of powder ( i ) for each ( i )</td>
<td>( a_{ij} ) for each ( i )</td>
<td>( \sum_{i=1}^{m} a_{ij} y_i )</td>
</tr>
</tbody>
</table>
Maximize producer’s revenue

The total revenue from sale of powders is \( \sum_{i=1}^{m} b_i y_i \), since the producer would sell \( b_i \) of powder \( i \) at a unit price of \( y_i \), for \( i = 1, \ldots, m \).

Thus, the producer solves the following problem to select prices:

\[
\begin{align*}
\max_y & \quad \sum_{i=1}^{n} b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad i = 1, \ldots, n \\
& \quad y_i \geq 0 \quad i = 1, \ldots, m
\end{align*}
\]

or equivalently

\[
\begin{align*}
\max_y & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \\
& \quad y \geq 0.
\end{align*}
\]
A dual pair of linear optimization problems

\[
\begin{align*}
\text{min}_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \\
\quad & \quad x \geq 0 \\
\text{max}_{y \in \mathbb{R}^m} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \\
\quad & \quad y \geq 0.
\end{align*}
\]

Notice how the roles of \( b \) and \( c \) have been reversed, and how \( A \) has been replaced by its transpose. The dimension of the variables has switched from the number of columns of \( A \) to its number of rows.

The optimal value of the maximization problem is the producer’s revenue from the sale of the powders to the consumer, which is of course also the total amount the consumer would spend on the nutrient powders.

Since the prices of the nutrient powders are chosen to undercut the prices of the foodstuffs, the optimal value of the maximization problem is no larger than the optimal value of the minimization problem. We will see later that the optimal values are the same.
Outline

1. The Diet Problem

2. Combinations of constraints
Getting a lower bound on the optimal value

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad x_1 + x_2 \\
\text{subject to} & \quad 3x_1 + x_2 \geq 7 \\
& \quad 2x_1 + x_2 \geq 6 \\
& \quad x_1 + 4x_2 \geq 10 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
Let $\bar{x}$ be a feasible solution.

Since $x_2 \geq 0$, the first constraint $3x_1 + x_2 \geq 7$ implies

$$3x_1 + 3x_2 \geq 7, \quad \text{so} \quad x_1 + x_2 \geq 2\frac{1}{3}. \quad \left(\text{Scale factor: } y_1 = \frac{1}{3}.\right)$$

We can get a slightly better bound by looking at the second constraint $2x_1 + x_2 \geq 6$ and exploiting $x_2 \geq 0$:

$$2x_1 + 2x_2 \geq 6 \quad \Rightarrow \quad x_1 + x_2 \geq 3 \quad \left(\text{Scale factor: } y_2 = \frac{1}{2}.\right)$$
Using combinations of constraints

To improve the bound further, we look at weighted combinations of the constraints.

Weights: \( y_1 = \frac{3}{11}, y_2 = 0, y_3 = \frac{2}{11} \).

Get:
\[
\frac{3}{11} (3x_1 + x_2 \geq 7) \quad + \quad \frac{2}{11} (x_1 + 4x_2 \geq 10) \quad \implies \quad x_1 + x_2 \geq 3\frac{8}{11}.
\]

There is a combination that gives the optimal value as a lower bound:

Weights: \( y_1 = 0, y_2 = \frac{3}{7}, y_3 = \frac{1}{7} \).

Get:
\[
\frac{3}{7} (2x_1 + x_2 \geq 6) \quad + \quad \frac{1}{7} (x_1 + 4x_2 \geq 10) \quad \implies \quad x_1 + x_2 \geq 4.
\]
Combinations of constraints

General linear combinations

We can represent the combined inequality as

\[ \sum_{i=1}^{m} y_i \left( \sum_{j=1}^{n} a_{ij} x_j \right) \geq \sum_{i=1}^{m} y_i b_i \quad \text{or equivalently} \quad \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j \geq b^T y \]

This inequality is satisfied by any feasible solution \( \bar{x} \) to the original problem provided the weights \( y \) are nonnegative.

Get a lower bound of \( b^T y \) on \( c^T x \) provided \( \sum_{i=1}^{m} a_{ij} y_i \leq c_j \) for \( j = 1, \ldots, n \) and \( y \geq 0 \), since \( x \geq 0 \).

In matrix terms, with weights \( y \geq 0 \) on the constraints, we need \( A^T y \leq c \) to be able to conclude that \( b^T y \) is a valid lower bound.
The dual problem

The dual problem is to maximize this lower bound:

\[
\begin{align*}
\max_y & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \\
& \quad y \geq 0.
\end{align*}
\]

We can write this out explicitly as:

\[
\begin{align*}
\max_{y \in \mathbb{R}^3} & \quad 7y_1 + 6y_2 + 10y_3 \\
\text{subject to} & \quad 3y_1 + 2y_2 + y_3 \leq 1 \\
& \quad y_1 + y_2 + 4y_3 \leq 1 \\
& \quad y_1, y_2, y_3 \geq 0.
\end{align*}
\]

Optimal solution is \( y = (0, \frac{3}{7}, \frac{1}{7})^T \), with value 4.
The primal and dual problems

Primal problem:

\[
\min_{x \in \mathbb{R}^2} \quad x_1 + x_2 \\
\text{subject to} \quad 3x_1 + x_2 \geq 7 \quad \text{Optimal solution is} \quad x = (2, 2), \\
2x_1 + x_2 \geq 6 \quad \text{with value 4} \\
x_1 + 4x_2 \geq 10 \\
x_1, x_2 \geq 0.
\]

Dual problem:

\[
\max_{y \in \mathbb{R}^3} \quad 7y_1 + 6y_2 + 10y_3 \\
\text{subject to} \quad 3y_1 + 2y_2 + y_3 \leq 1 \quad \text{Optimal solution is} \quad y = (0, \frac{3}{7}, \frac{1}{7})^T, \\
y_1 + y_2 + 4y_3 \leq 1 \quad \text{with value 4.} \\
y_1, y_2, y_3 \geq 0.
\]
Now a 3-dimensional example

\[
\begin{align*}
\text{min}_{x \in \mathbb{R}^3} & \quad 7x_1 + 5x_2 + 8x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + 5x_3 \geq 6 \\
& \quad 3x_1 + 2x_2 + 4x_3 \geq 13 \\
& \quad 2x_1 + x_2 + 2x_3 \geq 8 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

Let \( \bar{x} \) be feasible. We can combine the second and third constraints:

\[
\begin{align*}
7\bar{x}_1 + 5\bar{x}_2 + 8\bar{x}_3 & \geq 7\bar{x}_1 + 4\bar{x}_2 + 8\bar{x}_3 \\
= \quad (3\bar{x}_1 + 2\bar{x}_2 + 4\bar{x}_3) + 2(2\bar{x}_1 + \bar{x}_2 + 2\bar{x}_3) \\
\geq \quad 13 + 16 = 29.
\end{align*}
\]
The dual problem

Provided the scale factors are nonnegative, we get a valid lower bound. In particular, with weights $y \geq 0$ on the constraints, we need $A^T y \leq c$ to be able to conclude that $b^T y$ is a valid lower bound.

The dual problem is

$$\begin{align*}
\max_{y \in \mathbb{R}^3} & \quad 6y_1 + 13y_2 + 8y_3 \\
\text{subject to} & \quad y_1 + 3y_2 + 2y_3 \leq 7 \\
& \quad 2y_1 + 2y_2 + y_3 \leq 5 \\
& \quad 5y_1 + 4y_2 + 2y_3 \leq 8 \\
& \quad y_1, y_2, y_3 \geq 0.
\end{align*}$$
The primal and dual problems

Primal problem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^3} & \quad 7x_1 + 5x_2 + 8x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + 5x_3 \geq 6 \\
& \quad 3x_1 + 2x_2 + 4x_3 \geq 13 \\
& \quad 2x_1 + x_2 + 2x_3 \geq 8 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

Optimal solution is \( x = (3, 0, 1) \) with value 29.

Dual problem:

\[
\begin{align*}
\max_{y \in \mathbb{R}^3} & \quad 6y_1 + 13y_2 + 8y_3 \\
\text{subject to} & \quad y_1 + 3y_2 + 2y_3 \leq 7 \\
& \quad 2y_1 + 2y_2 + y_3 \leq 5 \\
& \quad 5y_1 + 4y_2 + 2y_3 \leq 8 \\
& \quad y_1, y_2, y_3 \geq 0.
\end{align*}
\]

Optimal solution is \( y = (0, 1, 2) \) with value 29.

This is strong duality: the optimal values agree (provided they exist).
Complementary slackness

Primal problem:

\[
\begin{align*}
\text{min}_{x \in \mathbb{R}^3} & \quad 7x_1 + 5x_2 + 8x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + 5x_3 \geq 6 \\
& \quad 3x_1 + 2x_2 + 4x_3 \geq 13 \\
& \quad 2x_1 + x_2 + 2x_3 \geq 8 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

At \( x = (3, 0, 1) \):
active constraint

Dual problem:

\[
\begin{align*}
\text{max}_{y \in \mathbb{R}^3} & \quad 6y_1 + 13y_2 + 8y_3 \\
\text{subject to} & \quad y_1 + 3y_2 + 2y_3 \leq 7 \\
& \quad 2y_1 + 2y_2 + y_3 \leq 5 \\
& \quad 5y_1 + 4y_2 + 2y_3 \leq 8 \\
& \quad y_1, y_2, y_3 \geq 0.
\end{align*}
\]

Optimal solution is \( y = (0, 1, 2) \)

Note also that at optimality, the **active constraints** in one problem correspond to the **positive components** in the other. This is an illustration of **complementary slackness**.