Math Models of OR: The Klee-Minty Cube

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Outline

1. Visiting all extreme points
2. The Klee-Minty cube in $\mathbb{R}^3$
3. The iterations
4. Alternative pivot rules
5. Extending to $\mathbb{R}^n$
6. Average performance
Worst-case performance of simplex

The simplex algorithm proceeds from one extreme point to a neighboring extreme point that is at least as good.

How many extreme points can it visit?

The Klee-Minty cube is an example where it visits every extreme point. It’s a problem that is expressed in terms of \( n \) inequality constraints on \( n \) variables, together with nonnegativity constraints. The number of extreme points is exponential in the size of the problem, \( 2^n \).

Thus, simplex has what is called exponential runtime in the worst case. Polynomial runtime (for example, \( 4n^3 \) iterations) is far preferable: it grows far more slowly than exponential runtime.

For large scale problems, linear run time (for example, \( 3n \) iterations) is desirable.
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The Klee-Minty cube in \( \mathbb{R}^3 \)

Consider the linear program

\[
\begin{align*}
\text{min } & \quad -100x_1 - 10x_2 - x_3 \\
\text{subject to } & \quad x_1 \leq 1 \\
& \quad 20x_1 + x_2 \leq 100 \\
& \quad 200x_1 + 20x_2 + x_3 \leq 10000 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]
The Klee-Minty cube in $\mathbb{R}^3$

The Klee-Minty cube

The Klee-Minty cube in $\mathbb{R}^3$
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Pivot rule

The variable that enters the basis is the nonbasic variable with the most negative reduced cost.

There are no ties in the minimum ratio for this example, so it is not necessary to specify the tie-breaking mechanism.
Get initial canonical form

After introducing slack variables, we have a problem in canonical form so we can proceed directly with simplex. The initial basis consists of the slack variables, denoted $x_4, x_5, x_6$.

<table>
<thead>
<tr>
<th>ratio</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
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After one iteration

\[ R_0 + 100R_1, R_2 - 20R_1, R_3 - 200R_1 \]

\[ \rightarrow \]

<table>
<thead>
<tr>
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<th>( x_1 )</th>
<th>( x_2 )</th>
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After two iterations

\[ R_0 + 10R_2, R_3 - 20R_2 \]

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After three iterations

\[ R_0 + 100R_1, R_2 + 20R_1, R_3 - 200R_1 \]

\[ \rightarrow \]

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<th>(x_1)</th>
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<th>(x_3)</th>
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</table>

Mitchell

The Klee-Minty Cube
After four iterations

\[
\frac{R_0 + R_3}{\rightarrow}
\]

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<th>( x_3 )</th>
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</table>
After five iterations

\[
R_0 + 100R_1, R_2 - 20R_1, R_3 + 200R_1
\]

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<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
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</table>
After six iterations

\[ R_0 + 10R_2, R_3 + 20R_2 \]

\[ \rightarrow \]

<table>
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<th>( x_2 )</th>
<th>( x_3 )</th>
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</tbody>
</table>
After seven iterations

\[
R_0 + 100R_1, R_2 + 20R_1, R_3 + 200R_2
\]

This is optimal, with value \(-10000\) and \(x_1 = 0, x_2 = 0, x_3 = 10000\).

The number of pivots required was \(7 = 2^3 - 1\).

Every extreme point was visited.
All the iterations

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Alternative pivot rules

If the best improvement rule had been used to choose the incoming variable, $x_3$ would have entered the basis on the first iteration and the problem would have been solved in one step.

A variant of the Klee-Minty cube has been designed that also requires exponentially many iterations in the worst-case, even with the best improvement rule.

Other variants with exponential worst-case performance have been designed for all known pivot rules.
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Extending to $n$ variables

\[ \begin{align*}
    n &= 3: \\ 
    \min_{x \in \mathbb{R}^n} & \quad -10^2 x_1 - 10 x_2 - x_3 \\ 
    \text{s.t.} & \quad x_1 \leq 1 \\ & \quad 20 x_1 + x_2 \leq 100 \\ & \quad 200 x_1 + 20 x_2 + x_3 \leq 10000 \\ & \quad x_j \geq 0 \quad \forall j
\end{align*} \]

\[ \begin{align*}
    \min_{x \in \mathbb{R}^n} & \quad - \sum_{j=1}^{n} 10^{n-j} x_j \\ 
    \text{subject to} & \quad 2 \sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \leq 100^{i-1} \\ & \quad x_j \geq 0 \quad \forall j = 1, \ldots, n
\end{align*} \]

The simplex method choosing the most negative entry to enter the basis requires $2^n - 1$ iterations to solve this problem.
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Average performance of simplex

There is theoretical analysis showing that the *average number of iterations is linear* in the size of the linear optimization problem.

(Need to make assumptions about the distribution of instances.)

This is also the *typical performance in practice.*