1 Introduction

Consider the linear programming problem

$$\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad 0 \leq x \leq u
\end{align*}$$

Could introduce slack variables $s$ to get standard form:

$$\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x + s = u \\
& \quad x, s \geq 0
\end{align*}$$

Instead, modify the pivot rules and work with the original tableau. A variable can be nonbasic at either its lower bound or its upper bound. If it is at its upper bound, it can enter the basis if its reduced cost is positive. The variable will be decreased. In the minimum ratio test, we ensure that no variable violates its upper or lower bound.

2 Example

$$\begin{align*}
\min & \quad -2x_3 - 3x_4 \\
\text{subject to} & \quad x_1 + x_3 + 3x_4 = 6 \\
& \quad x_2 - x_3 - 2x_4 = -1 \\
& \quad 0 \leq x_1 \leq 4 \\
& \quad 0 \leq x_2 \leq 3 \\
& \quad 0 \leq x_3 \leq 5 \\
& \quad 0 \leq x_4 \leq 1
\end{align*}$$

Have basic feasible solution:

Nonbasics: $x_3 = 0$, $x_4 = 1$
Basics: $x_1 = 6 - x_3 - 3x_4 = 3$, $x_2 = -1 + x_3 + 2x_4 = 1$.

Note how $x_4$ impacts these values.

Objective function value: $-2(0) - 3(1) = -3$. 
Since $x_4$ is at its upper bound and has a negative reduced cost, we don’t bring $x_4$ into the basis. Instead, $x_3$ enters the basis.

Calculate the simplex direction:

We are increasing $x_3$, so the changes in the basic variables are given by the negatives of the entries in the $x_3$ column of the tableau.

Direction is $\Delta x = (-1, 1, 1, 0)^T$.

3 Minimum Ratio Test

Taking a step of length $t$ in the simplex direction gives a new point:

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} = 
\begin{bmatrix}
  3 \\
  1 \\
  0 \\
  1
\end{bmatrix} + t 
\begin{bmatrix}
  -1 \\
  1 \\
  1 \\
  0
\end{bmatrix}
$$

Need to keep $x$ within its bounds.

- $x_1$ is decreasing. Need $t \leq 3$ to keep $x_1 \geq 0$.
- $x_2$ is increasing. Need $t \leq 2$ to keep $x_2 \leq 3$.
- $x_3$ is increasing. Need $t \leq 5$ to keep $x_3 \leq 5$.

So choose $t = 2$. $x_2$ leaves the basis at its upper bound.
4 Tableaus

1. \[ \begin{array}{cccc|cc}
    x_1 & x_2 & x_3 & x_4 & \text{Min ratios} \\
    \hline
    0 & 0 & -2 & -3 & \text{LB} & \text{UB} \\
    6 & 1 & 0 & 1 & 3 & \text{LB} & \text{UB} \\
    -1 & 0 & 1 & 2 & -2 & \text{LB} & \text{UB} \\
    \hline
    \text{UB} & 4 & 3 & 5 & 1 & \\
\end{array} \]

B/LB/UB \hspace{0.5cm} B \hspace{0.5cm} B \hspace{0.5cm} LB \hspace{0.5cm} UB

Values \hspace{0.5cm} 3 \hspace{0.5cm} 1 \hspace{0.5cm} 0 \hspace{0.5cm} 1

Initial value of objective is \( z = 0 - 3(1) = -3 \).

\( x_3 \) enters the basis. The “Min ratios” columns correspond to variables leaving the basis: if the pivot is in the first constraint, \( x_1 \) would leave the basis at its lower bound, and if the pivot is in the second constraint, \( x_2 \) would leave the basis at its upper bound. By the minimum ratio test, \( x_2 \) leaves the basis at its upper bound.

2. \[ -R_2 \text{ then } R_0 + 2R_2, \ R_1 - R_2 \]

\[ \begin{array}{cccc|cc}
    x_1 & x_2 & x_3 & x_4 & \text{Min ratios} \\
    \hline
    2 & 0 & -2 & 0 & 1 & \text{LB} & \text{UB} \\
    5 & 1 & 1 & 0 & 1 & \text{LB} & \text{UB} \\
    1 & 0 & -1 & 1 & 2 & \text{LB} & \text{UB} \\
    \hline
    \text{UB} & 4 & 3 & 5 & 1 & \\
\end{array} \]

B/LB/UB \hspace{0.5cm} B \hspace{0.5cm} UB \hspace{0.5cm} B \hspace{0.5cm} UB

Values \hspace{0.5cm} 1 \hspace{0.5cm} 3 \hspace{0.5cm} 2 \hspace{0.5cm} 1

Value of objective is \( z = -2 - 2(3) + 1(1) = -7 \).

Basics: \( x_1 = 5 - x_2 - x_4 = 1, x_3 = 1 + x_2 - 2x_4 = 2 \).
$x_4$ enters the basis and decreases. Thus, $x_1$ and $x_3$ increase.

Simplex direction:

We are decreasing $x_4$, so the changes in the basic variables are given by the entries in the $x_4$ column of the tableau. Direction is $\Delta x = (1, 0, 2, -1)^T$.

The value of the minimum ratio for the basic variables is 2.5, which would drive $x_3$ to its upper bound.

The incoming basic variable also provides an upper bound on the maximum possible step length. Since we require $x_4 \geq 0$, the step length must be $\leq 1$.

Thus, the pivot is to keep $x_4$ nonbasic, but switch it from being nonbasic at its upper bound to nonbasic at its lower bound. This changes the value of the basic variables $x_1$ and $x_3$.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Min ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>UB</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

B/LB/UB   | $B$ | $UB$ | $B$ | LB |
Values 2   | 3   | 4    | 0   |

This tableau is in **optimal form**.

Optimal value of objective is $z = -2 - 2(3) + 1(0) = -8$.

Basics: $x_1 = 5 - x_2 - x_4 = 2$, $x_3 = 1 + x_2 - 2x_4 = 4$.

$x_2$ is at its upper bound, and decreasing $x_2$ will increase the objective function value. $x_4$ is at its lower bound, and increasing $x_4$ will increase the objective function value.