Math Models of OR: Convexity

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Convex sets

Recall:

**Definition**

Let \( x \) and \( y \) be two points in \( \mathbb{R}^n \). The **line segment** joining them is the set of points lying on the straight line between them and is denoted \([x, y]\).

We use this definition to define convex sets:

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Let \( S \subseteq \mathbb{R}^n \). The set \( S \) is convex if and only if:

for any two points \( x \in S \), \( y \in S \), the line segment \([x, y]\) joining them is contained within \( S \).
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Some convex sets

convex

not convex

convex
Examples of convex sets

Hyperplanes, halfspaces, and polyhedra are all convex.

A set consisting of a single point is convex: we can’t get two distinct points in $S$, so the definition holds vacuously.

Similarly, the empty set is convex.

Balls and ellipsoids are also convex.

Convex sets can be unbounded.
Convex combinations

A more algebraic way of expressing the line segment between two points $x$ and $y$:

**Definition**

A **convex combination** of $x$ and $y$ is a point of the form $w = \lambda x + (1 - \lambda)y$ with the scalar $\lambda$ satisfying $0 \leq \lambda \leq 1$. The set of all convex combinations of $x$ and $y$ is the line segment between $x$ and $y$. 

\[ \lambda x + (1-\lambda)y = y + \lambda (x-y) \]
Convexity of polyhedra

Consider the standard form linear optimization problem:

$$\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}$$

**Theorem**

The feasible region of the standard form linear optimization problem is convex.

For any two feasible points, every point on the line segment joining them is feasible.
Proof, part 0

We need to consider two generic feasible points $\bar{x}$ and $\bar{y}$, and show that every point on the line segment between them is also feasible.

Equivalently, we need to show that for any feasible $\bar{x}$ and $\bar{y}$ and any scalar $\lambda$ satisfying $0 \leq \lambda \leq 1$, the point

$$w := \lambda \bar{x} + (1 - \lambda) \bar{y}$$

is feasible.

Assume: $A\bar{x} = b, \bar{x} \geq 0$ $A\bar{y} = b, \bar{y} \geq 0$

Show: $A\omega = b, \omega \geq 0$
Proof, part 1

Show $Aw = b$:

We have

$$
Aw = A(\lambda \bar{x} + (1 - \lambda)\bar{y}) \\
= \lambda A\bar{x} + (1 - \lambda)A\bar{y} \\
= \lambda b + (1 - \lambda)b \text{ since } \bar{x}, \bar{y} \text{ feasible} \\
= b
$$

so $A\bar{x} = b, A\bar{y} = b$

as required.
Proof, part 2

Show $w \geq 0$:

We have

$$w = \lambda \bar{x} + (1 - \lambda)\bar{y} \geq 0 \quad \text{since } \bar{x} \geq 0, \bar{y} \geq 0, 0 \leq \lambda \leq 1$$

as required.

Hence, for any two feasible points, the line segment joining them is contained in the feasible region.
Convexity of optimal set

This result can be extended straightforwardly to show that the feasible region of any linear optimization problem is convex.

Further, we can say something about the set of optimal solutions.

**Theorem**

*The set of optimal solutions to a standard form linear optimization problem is convex.*
Proof

First, note that if the problem is infeasible or has an unbounded optimal value then the set of optimal solutions is the empty set, which is a convex set.

If the LOP has a finite optimal value $z$ then the set of optimal solutions is the set of feasible solutions to the standard form linear optimization problem

$$\min_{x \in \mathbb{R}^n} \quad 0$$

subject to

$$Ax = b$$

$$c^T x = z$$

$$x \geq 0$$

By the first Theorem, the set of feasible solutions to this problem is convex.
Convex optimization problems

Nonlinear optimization problems that are convex are typically easier to solve than nonconvex ones.

They possess the nice property that any local minimizer is a global minimizer.

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is convex if \( \{(x, y) \subseteq \mathbb{R}^{n+1} : y \geq f(x)\} \) is convex; equivalently, any tangent line to the function stays below the function.
Other convex functions:

\[ y = x^2 \]

E.g.: \[ y = x^2 \]

Not all quadratic functions are convex.

E.g.: \[ y = -x^2 \]