Math Models of OR: Extreme Points and Basic Feasible Solutions

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Outline

1. Introduction

2. Let \( \bar{x} \) be a basic feasible solution, show it is an extreme point

3. Let \( \bar{x} \) be an extreme point, show it is a basic feasible solution
   - Example
   - The general case
Extreme points

Recall that if a standard form linear optimization problem

$$\min_{x \in \mathbb{R}^n} \quad c^T x$$
subject to \quad $Ax = b$
$$x \geq 0$$

has an optimal solution then it has an optimal solution that is an extreme point.

**Definition**

Let $P$ be the feasible set of a linear program. A point $\bar{x} \in P$ is an extreme point of $P$ if it is not on the line segment joining two other points in $P$. 
Basic feasible solutions

The simplex algorithm works with canonical form tableaus and moves from basic feasible solution to adjacent basic feasible solution.

The **basic feasible solution** corresponding to a canonical form tableau is obtained by setting the nonbasic variables equal to zero and then finding the unique solution for the basic variables.
The relationship

We’ve seen graphically that basic feasible solutions correspond to extreme points. We prove the following theorem to make this formal:

**Theorem**

Let $\bar{x}$ be a feasible solution to a standard form linear optimization problem. The point $\bar{x}$ is a basic feasible solution if and only if it is an extreme point of the feasible region of the linear optimization problem.

The theorem is an “if and only if” theorem, so we need to show two directions.
For example

\[ \begin{align*}
    x_1 &= 3, \\
    x_4 &= 0
\end{align*} \]

Optimal contour:

\[ -x_1 - 2x_2 = -5 \]

\[ x_2 = 2, x_5 = 0 \]

\[ \min_{x \in \mathbb{R}^2} -x_1 - 2x_2 \]

subject to

\[ \begin{align*}
    x_1 + 3x_2 &\leq 6 \\
    x_1 &\leq 3 \\
    x_2 &\leq 2 \\
    x_1, x_2 &\geq 0
\end{align*} \]
Let $\bar{x}$ be a basic feasible solution, show it is an extreme point

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Assume $\bar{x}$ is not an extreme point and derive a contradiction

Since $\bar{x}$ is assumed to not be extreme, it must be the midpoint between two other distinct feasible points $x^{(1)}$ and $x^{(2)}$, so $\bar{x} = \frac{1}{2}x^{(1)} + \frac{1}{2}x^{(2)}$.

This means that for each component $j = 1, \ldots, n$, we have

$$\bar{x}_j = \frac{1}{2}x^{(1)}_j + \frac{1}{2}x^{(2)}_j.$$

$x^{(1)} = (1, 4)$

$x^{(2)} = (11, 2)$

$\bar{x} = (6, 3)$

line segment
Let $\bar{x}$ be a basic feasible solution, show it is an extreme point.

Nonbasic components of $\bar{x}$

We look in particular at nonbasic components of $\bar{x}$, so $\bar{x}_j = 0$.

For these components, we have

$$
\bar{x}_j = \frac{1}{2} x_j^{(1)} + \frac{1}{2} x_j^{(2)}
$$

$= 0$ for nonbasic components $\geq 0$ since $x^{(1)}$ feasible $\geq 0$ since $x^{(2)}$ feasible

It follows that we must have $x_j^{(1)} = x_j^{(2)} = 0$ for all the nonbasic components of $\bar{x}$.

But once the nonbasic components are fixed, the values of the basic variables are then uniquely determined: they are equal to the corresponding entries in $b$.

So we must have $x^{(1)} = x^{(2)} = \bar{x}$. So the points $x^{(1)}$ and $x^{(2)}$ are not distinct, so $\bar{x}$ is actually an extreme point.
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

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Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

Introduction to proof

This direction is harder to prove, and the proof needs some linear algebra.

We prove it by contradiction, so we assume $\bar{x}$ is not a BFS and show it is not an extreme point.

To make the proof a little simpler, we assume $\bar{x}$ has exactly $m$ positive components; the proof can be extended to handle the general case.
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Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

An example

We first work an example before abstracting the proof.

$$\min_{x \in \mathbb{R}^6} \quad 3x_1 + 2x_2 + x_3$$
subject to
$$x_1 + x_2 + x_3 + x_4 = 6$$
$$2x_1 + x_3 + x_5 = 4$$
$$-x_1 + x_2 + x_6 = 2$$
$$x_1, \ldots, x_6 \geq 0$$

The point $\bar{x} = (1, 3, 2, 0, 0, 0)$ is feasible. Is it a BFS? The tableau is

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Let \( \bar{x} \) be an extreme point, show it is a basic feasible solution

Two pivots

\[
\begin{array}{ccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
  0 & 3 & 2 & 1 & 0 & 0 & 0 \\
  6 & 1 & 1 & 1 & 1 & 0 & 0 \\
  4 & 2 & 0 & 1 & 0 & 1 & 0 \\
  2 & -1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[
R_0 - 2R_3, R_1 - R_3
\]

\[
\begin{array}{ccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
  -4 & 5 & 0 & 1 & 0 & 0 & -2 \\
  4 & 2 & 0 & 1 & 0 & 1 & -1 \\
  4 & 2 & 0 & 1 & 0 & 1 & 0 \\
  2 & -1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[
R_0 - R_1, R_2 - R_1
\]

\[
\begin{array}{ccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
  -6 & 3 & 0 & 0 & -1 & 0 & -3 \\
  4 & 2 & 0 & 1 & 1 & 0 & -1 \\
  0 & 0 & 0 & -1 & 1 & -1 \\
  2 & -1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

We don’t have a BFS

\[
\begin{array}{ccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
  -6 & 3 & 0 & 0 & -1 & 0 & -3 \\
  4 & 2 & 0 & 1 & 1 & 0 & -1 \\
  0 & 0 & 0 & -1 & 1 & -1 \\
  2 & -1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

The second constraint **does not involve** $x_1$, $x_2$, or $x_3$.

So our given feasible solution is **not a bfs**.
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

**Example**

A little linear algebra

Equivalently, we can say that this means **the original first three columns of $A$ are linearly dependent**.

In fact, notice that

$$
\begin{bmatrix}
1 & 1 & 1 \\
2 & 0 & 1 \\
-1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
-2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
$$

Because of this equality, we have a **direction we can move in** and still maintain feasibility.
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution.

The direction

Let $d = (1, 1, -2, 0, 0, 0)^T \in \mathbb{R}^6$. Then

$$\hat{x} := \bar{x} + td$$

satisfies $A\hat{x} = b$ for any $t$, positive or negative, since $A\bar{x} = b$ and $Ad = 0$.

Further, provided $-1 \leq t \leq 1$, we get $\hat{x} \geq 0$. 
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution.

**Example**

**Two new feasible points**

Want to show $\bar{x}$ is not extreme.

Let’s take the two new feasible points $x^{(1)}$ and $x^{(2)}$ with $t = 1$ and $t = -1$, respectively (the $x_4$, $x_5$, and $x_6$ components of $d$, $x^{(1)}$, $x^{(2)}$ and $\bar{x}$ are all zero, so we don’t write them out explicitly):

$$x^{(1)} = \bar{x} + d = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$x^{(2)} = \bar{x} - d = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

$\bar{x}$ is not extreme

Notice that

$$\frac{1}{2} x^{(1)} + \frac{1}{2} x^{(2)} = \frac{1}{2} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \bar{x}.$$ 

So $\bar{x}$ is not an extreme point.
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution.

Example

Graph this example

\[x_1 + x_2 + x_3 = 6, \quad x_4 = 0\]
Let \( \bar{x} \) be an extreme point, show it is a basic feasible solution.

Example

Graph this example

\[ x_1 + x_2 + x_3 = 6, \quad x_4 = 0 \]

\[ -x_1 + x_2 = 2, \quad x_6 = 0 \]
Let \( \bar{x} \) be an extreme point, show it is a basic feasible solution.

**Example**

Graph this example

\[\begin{align*}
x_1 + x_2 + x_3 &= 6, \quad x_4 = 0 \\
x_1 - x_2 &= 2, \quad x_6 = 0
\end{align*}\]
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution.

Example

Graph this example:

$-x_1 + x_2 = 2, x_6 = 0$

$2x_1 + x_3 = 4, x_5 = 0$

$x_1 + x_2 + x_3 = 6, x_4 = 0$
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution.

**Example**

Graph this example:

- $-x_1 + x_2 = 2, x_6 = 0$
- $2x_1 + x_3 = 4, x_5 = 0$
- Portion of feasible region with $x_4 = x_5 = x_6 = 0$

Points:
- $(0, 2, 4) = x^{(2)}$
- $(1, 3, 2) = \bar{x}$
- $(2, 4, 0) = x^{(1)}$

Equations:
- $x_1 + x_2 + x_3 = 6, x_4 = 0$
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

**Linear algebra characterization of BFS**

The example illustrates the following result:

Let $\bar{x}$ be a feasible solution to a standard form linear program with $m \times n$ constraint matrix $A$ with rank equal to $m$.

The point $\bar{x}$ is a basic feasible solution if and only if the columns of $A$ corresponding to the positive components of $\bar{x}$ are linearly independent.
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Return to the general case

Recall, we assume $\bar{x}$ is not a basic feasible solution and try to derive a contradiction.

Without loss of generality, we can assume the $m$ positive components of $\bar{x}$ are the first $m$ components, by renumbering the components if necessary. So we have

$$\bar{x}_j \begin{cases} > 0 & \text{for } j = 1, \ldots, m \\ = 0 & \text{for } j = m + 1, \ldots, n \end{cases}$$
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

The general case

### Row reducing $A$

We can write the constraint matrix $A$ as

$$A = \begin{bmatrix} \underbrace{B}_{m \text{ columns}} & \underbrace{N}_{(n-m) \text{ columns}} \end{bmatrix}$$

If $\bar{x}$ was a basic feasible solution, we would be able to use elementary row operations to turn this into a canonical form, so

$$A = [B \quad N] \rightarrow \begin{bmatrix} I & \hat{N} \end{bmatrix}$$

for some $m \times (n-m)$ matrix $\hat{N}$, where $I$ is the $m \times m$ identity matrix.
Let $\bar{x}$ be an extreme point, show it is a basic feasible solution

**But we don’t have a BFS**

Since $\bar{x}$ is not a basic feasible solution, this row reduction cannot be possible. That means the columns of $B$ are *linearly dependent*, which is equivalent to stating that

\[
\text{there exists a vector } d^B \in \mathbb{R}^m \text{ with } Bd^B = 0,
\]

\[
\text{where } d^B \text{ has at least one nonzero component.}
\]

We can extend $d^B$ out to a vector $d \in \mathbb{R}^n$ by appending zeroes:

\[
\text{let } d_j = \begin{cases} d^B_j & \text{for } j = 1, \ldots, m \\ 0 & \text{for } j = m + 1, \ldots, n \end{cases}
\]

so

\[
d = \begin{bmatrix} d^B \\ 0 \end{bmatrix}
\]

$m$ components

$(n - m)$ components
Moving in our direction

Notice that

\[ Ad = [B \quad N] \begin{bmatrix} dB \\ 0 \end{bmatrix} = BdB = 0, \]

so for any scalar \( t \) we have

\[ A(\bar{x} + td) = A\bar{x} + tAd = b + 0 = b. \]

Further, since \( d_j = 0 \) if \( \bar{x}_j = 0 \), we have \( \bar{x} + td \geq 0 \) for \( t \) sufficiently close to zero (positive or negative).

Thus, choose \( \bar{t} > 0 \) so that both \( \bar{x} + \bar{t}d \) and \( \bar{x} - \bar{t}d \) are feasible.

Then notice that \( \bar{x} \) is the midpoint of these two feasible points:

\[ \bar{x} = \frac{1}{2} (\bar{x} + \bar{t}d) + \frac{1}{2} (\bar{x} - \bar{t}d). \]

Thus, \( \bar{x} \) is **not an extreme point**. Thus, we’ve proved the contrapositive, so we do indeed have the result that if \( \bar{x} \) is an extreme point then it is a basic feasible solution.