1 Motivation

At each iteration, we move from a basic feasible solution to a neighboring basic feasible solution. When we perform a simplex pivot, we update every entry in the tableau. **Do we need to know all these entries in order to find the next BFS?**

We need to know:

- **the pivot column**, so we have to find a variable that has a negative cost. If the current tableau is in optimal form, we need to determine that by looking at every reduced cost.

- **the pivot row**, so we need to compute the minimum ratio. This requires knowledge of the entries in \( b \) and the pivot column. **Note:** once we’ve used the minimum ratio to select the pivot row, we only need to know two entries in that row in order to update \( x \): the value on the right hand side, and the value in the pivot column.

- **the basic sequence**, so we know which variable is leaving the basis.

For example, consider the tableau, with the circled pivot entry:

\[
\begin{array}{cccccc}
\text{ratio} & x_1 & x_2 & x_3 & x_4 & x_5 \\
5 & 0 & -3 & -3 & 0 & 0 & -1 \\
3 & 5 & 1 & 1 & 1 & 0 & 2 \\
3 & 3 & 0 & 0 & 1 & 4
\end{array}
\]

basic sequence \( S = (3, 4) \).

To determine the pivot position, we need to know the objective function costs, the right hand side \( b \), and the entries in the pivot column:

\[
\begin{array}{cccccc}
\text{ratio} & x_1 & x_2 & x_3 & x_4 & x_5 \\
0 & 0 & -3 & -3 & 0 & 0 & -1 \\
5 & 5 & 1 & * & * & * & * \\
3 & 3 & 1 & * & * & * & *
\end{array}
\]

basic sequence \( S = (3, 4) \).

We can tell from these entries that \( x_1 \) will replace \( x_4 \) in the basic sequence, since \( x_4 \) is the basic variable corresponding to the pivot row. We can also calculate the updated entries in \( b \), so we can calculate the new solution \( x \) based just on this part of the tableau:

\[
\begin{array}{cccccc}
R_0 + 3R_2, & R_1 - R_2 \\
\rightarrow & \\
9 & 0 & * & * & * & * \\
2 & 0 & * & * & * & * \\
3 & 1 & * & * & * & *
\end{array}
\]

basic sequence \( S = (3, 1) \).
So the updated basic feasible solution has basic variables \( x_3 = 2 \) and \( x_1 = 3 \), with nonbasics \( x_2 = x_4 = x_5 = 0 \) and value \(-9\).

In order to perform the next pivot, we need to be able to calculate the updated objective function terms efficiently. A note on terminology: the updated objective function entries \( c_j \) are the **reduced costs**. Thus, the entries in the top row of a simplex tableau are known as the reduced costs of the corresponding variables.

**2 Pivot matrices**

Pivoting corresponds to premultiplying the tableau by a **pivot matrix**. We find the pivot matrix using the simplex golden rule:

\[
\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}
\]

Let’s call the original tableau \( M_0 \). If we premultiply \( M_0 \) by \( Q_1 \), we get a new tableau:

\[
\begin{align*}
Q_1M_0 &= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3 & -3 & 0 & 0 & -1 \\ 5 & 1 & 1 & 0 & 2 \\ 3 & 1 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & -6 & 0 & 3 & 11 \\ 2 & 0 & 2 & 1 & -1 & -2 \\ 3 & 1 & -1 & 0 & 1 & 4 \end{bmatrix} =: M_1
\end{align*}
\]

The resulting tableau \( M_1 \) is identical to the one obtained by performing the pivot on the original tableau:

\[
\begin{array}{cccccc}
\text{ratio} & x_1 & x_2 & x_3 & x_4 & x_5 \\
0 & -3 & -3 & 0 & 0 & -1 \\
5 & 1 & 1 & 1 & 0 & 2 \\
3 & 1 & -1 & 0 & 1 & 4 \\
\end{array}
\]

Why do the two approaches agree? Let’s look at just one entry, \( a_{15} \). In the original pivot, we first divide \( a_{25} \) by the pivot entry (which is just \( a_{21} = 1 \) in this case), and then we subtract \( a_{11} \) multiplied by \( a_{25} \) from \( a_{15} \). Algebraically, we can write this update to \( a_{15} \) as

\[
a_{15} \leftarrow a_{15} - a_{11}a_{25}/a_{21}
\]

The pivot matrix \( Q_1 \) can also be expressed in terms of the entries in the pivot column:

\[
Q_1 = \begin{bmatrix} 1 & 0 & -c_1/a_{21} \\ 0 & 1 & -a_{11}/a_{21} \\ 0 & 0 & 1/a_{21} \end{bmatrix}
\]
The updated $x_5$ column of the product $Q_1M_0$ is equal to the matrix-vector product of $Q_1$ with the $x_5$ column of $M_0$:

$$\text{last column of } Q_1M_0 \text{ is } \begin{bmatrix} 1 & 0 & -c_1/a_{21} \\ 0 & 1 & -a_{11}/a_{21} \\ 0 & 0 & 1/a_{21} \end{bmatrix} \begin{bmatrix} c_5 \\ a_{15} \\ a_{25} \end{bmatrix} = \begin{bmatrix} * \\ a_{15} - a_{25}a_{11}/a_{21} \\ * \end{bmatrix}$$

exactly the same formula as calculated for the pivot update. All the other entries work similarly.

We can now perform another iteration on the updated tableau $M_1$. The new pivot column would be the $x_2$ column and the pivot would:

(i) divide row 1 by $2$, $\frac{1}{2}R_1$,
(ii) add 6 copies of row 1 to row 0, $R_0 + 6R_1$,
(iii) add 1 copy of row 1 to row 2, $R_2 + R_1$.

Performing these operations on the identity matrix gives the pivot matrix $Q_2$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{2}R_1 \text{ then } R_0 + 6R_1, R_2 + R_1 \rightarrow Q_2 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$

If we premultiply $M_1$ by $Q_2$, we get a new tableau:

$$Q_2M_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & -6 & 0 & 3 & 11 \\ 2 & 0 & 2 & 1 & -1 & -2 \\ 3 & 1 & -1 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 15 & 0 & 0 & 3 & 0 & 5 \\ 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -1 \\ 4 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 3 \end{bmatrix} =: M_2$$

### 3 The revised simplex method

The revised simplex method carries out exactly the pivots of the usual simplex method, but uses pivot matrices to calculate required entries, and is selective about which entries get calculated.

Note that since we never pivot on the objective function row, the first column of the pivot matrix is always the first column of the identity matrix.

We give an example of solving a linear optimization problem with the revised simplex method in a separate handout.
4 A symbolic representation of the pivot matrix and the tableau

We have an initial tableau which we can write as

\[
M_0 = \begin{bmatrix}
-d & c^T \\
0 & b \\
-\hat{d} & \hat{c}^T \\
\hat{b} & \hat{A}
\end{bmatrix}
\]

After a few pivots, we end up with a new canonical form tableau

\[
\hat{M} = \begin{bmatrix}
-\hat{d} & \hat{c}^T \\
\hat{b} & \hat{A}
\end{bmatrix}
\]

The sequence of pivots can be represented by a single pivot matrix \( P \). So we have \( \hat{M} = PM_0 \).

Let us assume without loss of generality that the basic variables consist of the first \( m \) columns of \( \hat{M} \), by reordering the variables if necessary. So we can write

\[
\hat{M} = \begin{bmatrix}
-\hat{d} & 0^T & \hat{c}_N^T \\
\hat{b} & I & \hat{N}
\end{bmatrix}
\]

The original tableau can be similarly written

\[
M_0 = \begin{bmatrix}
-d & c^T_b & c_N^T \\
0 & b & B & N
\end{bmatrix}
\]

We also choose to write the pivot matrix as

\[
P = \begin{bmatrix}
1 & -y^T \\
0 & G
\end{bmatrix}
\]

for some vector \( y \in \mathbb{R}^m \) and matrix \( G \in \mathbb{R}^{m \times m} \). The equation \( \hat{M} = PM_0 \) can then be written

\[
\hat{M} = \begin{bmatrix}
-\hat{d} & 0^T & \hat{c}_N^T \\
\hat{b} & I & \hat{N}
\end{bmatrix} = PM_0 = \begin{bmatrix}
1 & -y^T \\
0 & G
\end{bmatrix} \begin{bmatrix}
-d & c^T_b & c_N^T \\
0 & b & B & N
\end{bmatrix} = \begin{bmatrix}
-d - y^T b & c^T_b - y^T B & c_N^T - y^T N \\
G b & GB & GN
\end{bmatrix}
\]

Comparing the entries in the portion of the constraint matrix corresponding to the basic variables shows that

\[
I = GB,
\]

so we must have

\[
G = B^{-1},
\]

the inverse of the matrix \( B \). From the objective function coefficients for the basic variables, we obtain

\[
0^T = c^T_b - y^T B, \quad \text{so } y^T = c^T_b B^{-1}.
\]

We will give interpretations to \( y \) and the other terms in \( \hat{M} \) when we discuss duality.