Homework 4 posted; due Wednesday, May 7.

We’ll begin with Poisson point processes in one dimension which actually are an example of both a Poisson point process and a renewal process.

One convenient way to define a Poisson point process in one dimension is the set of moments in time where the Poisson counting process changes state.

![Diagram of Poisson point process]

We can label the Poisson point process as the sequence of times generated in this way: \( t_j = \inf\{t > 0: X(t) = j\} \). The statistics of these times (which are random) can be most conveniently characterized in terms of the interval between successive times: 

\[ T_j = t_j - t_{j-1}. \]

Using the results we computed from the Poisson counting process and mapping them to the Poisson point process, we have the following characterizing property of the Poisson point process:

- \( \{T_j\}_{j=1}^\infty \) are iid random variables which are exponentially distributed with mean \( 1/\lambda \) where \( \lambda \) was the "birth rate" in the associated Poisson counting process.

When referring to the Poisson point process, the parameter \( \lambda \) is referred to as the intensity (or rate or density) of the Poisson point process.

- The reason for this terminology is that we can alternatively view point processes in terms of how many points are generated in a fixed interval \([a, b]\). For the Poisson point process, the number of points in such an interval is a Poisson random variable with mean \( \lambda(b - a) \), so we see that \( \lambda \) is the average density of the Poisson point process.

- Also, one can show (again by referring to the Poisson counting process associated to the Poisson point process) that the number of Poisson points in nonoverlapping (Borel) sets is independent. This follows from the Markov property of the Poisson counting process (future and past is conditionally independent, given the present, and choose the present moment to be "between" the two sets for the case those sets are intervals.)

Poisson point processes are widely used in modeling the times of random incidents in dynamical models:

- Arrivals of demand in a queueing model
• Arrival of cars at an intersection or an entry ramp to an expressway
• Arrival of input signals to neurons in a network
• Times at which claims are filed at an insurance company

More broadly, Poisson point processes turn out to be good models for systems where the incidents can be viewed as a superposition of a large number of processes which generate incidents rarely, and are independent of each other.

• this is the modeling interpretation of the Poisson limit theorem
  ○ see Karlin and Taylor Sec. 5.9
• Queueing model: The demands are arising from a large pool of noninteracting potential clients who each are making rare demands
  ○ Seasonal variations (i.e., holidays, sales, etc.) can actually still be accounted for by a nonhomogenous Poisson point process which has an intensity $\lambda(t)$ that depends on time $t$. The above formulas need to be modified, i.e., with hazard functions for the time delay between successive incidents, and the number of points in an interval $[a, b]$ is Poisson distributed with mean $\int_a^b \lambda(t) \, dt$.

• Traffic model: Independent large number of cars except in cases like:
  ○ regulation by traffic lights or police shortly before the entrance
  ○ random accident causes many cars to divert behavior
  ○ rush hour and other regular variations in traffic behavior can be modeled as nonhomogenous Poisson point process
  ○ funerals and other coordinated random traffic activity

• Neuronal networks: Violations of Poisson point process model when inputs are only determined by a small number of neurons upstream and/or synchronization upstream
  ○ some statistics look robustly like Poisson point process while others (in frequency domain) do not
    • http://journals.aps.org.libproxy.rpi.edu/pre/abstract/10.1103/PhysRevE.73.022901

• Insurance claims:
  ○ large-scale disasters would violate the assumption because it's an unpredictable source to generate many claims together.

Multidimensional Poisson Point Processes

To make this extension, we will characterize the properties of the one-dimensional point process within a Poisson random measure framework.

$N(A) = \#$ Poisson points in a Borel set $A$.

(As usual in measure theory, one has to prescribe a suitably cooperative $\sigma$–algebra of sets to work with, and we’ll take Borel sets).
1. Countable additivity (true of all measures)

\[ N(A) = \sum_{j=1}^{\infty} N(B_j) \] if A is the disjoint union of \( \{B_j\}_{j=1}^{\infty} \). Also true of finite unions.

2. For Poisson random measure, under the above relationship between A and \( \{B_j\}_{j=1}^{\infty} \), \( N(B_j) \) are independent of each other.

3. The mean number of Poisson points in a set A is

\[ \mathbb{E}[N(A)] = \lambda \ell(A) \]

where \( \ell(A) \) is the Lebesgue measure of set A, which is just a generalization of the notion of length.

These properties are actually sufficient to uniquely prescribe the Poisson point process in one dimension (alternative to the prescription regarding the time intervals between incidents). Note that we did not have to assume that the number of points in a given set are given by the Poisson distribution. That is in fact a consequence of the assumptions above, as we'll argue a little later.

Proceeding for now with the practical side of Poisson point processes, the importance of the above characterization is that it extends to multiple dimensions (unlike the characterization in terms of interincident times).

To generate a multidimensional Poisson point process in practice, one can show that the basic assumptions imply that you should do the following: Organize your simulation or (Monte Carlo) calculation so that it is referring to disjoint sets with finite measure (length/volume/area).

For each disjoint region \( B_j \) generate a Poisson random variable for the number of points in that region:
$N(B_j) \sim \text{Poi}(\lambda \ell(B_j))$ and these random variables are independent across disjoint regions. Then distribute these points uniformly over that region, with the location of each point independent of the location of other points.

- How simulate this for irregular regions? Use rejection method which is a way of using sampling of simple probability distributions to sample from complicated probability distributions. To employ it, one has to choose a simple probability distribution that "bounds" the complicated probability distribution.
  - Enclose the domain $B_j$ in some cuboid (multidimensional rectangle).
  - Generate random uniform points in the cuboid. If it falls within the set $B_j$ then accept the point, otherwise, reject and redo.
  - One can show by a conditional probability calculation that this will generate points uniformly in the set $B_j$.

Note the rejection method also can be used to sample from complicated probability distributions by sampling with rejection from simpler probability distributions. Note sampling from a PDF is choosing a uniform random point under the graph of the PDF!

Applications of multidimensional Poisson point process are typically to generate random locations in space with density/intensity $\lambda$.

- defects in materials
- vortices in fluid
- food sources in an ecosystem
- baseline for distribution of stars/galaxies

Why is the Poisson distribution associated with simple point processes? Because the Poisson distribution is the simplest discrete probability distribution that has the property of being infinitely divisible.

Let's see what the defining property of the multidimensional Poisson point process implies about the random variable characterizing the number of points in sets.
And in fact, this relationship must remain true if I choose finer and finer subdivisions of the form

\[ A = \bigcup_{j=1}^{m} B_j \]

This is what is meant by infinite divisibility of a random variable, that it can be expressed as a sum of arbitrarily many independent random variables. This is a restrictive condition, as I'll argue in a minute, and is satisfied by some special probability distributions, among which are Poisson and Gaussian (continuous), but not binomial. How can you actually check whether a probability distribution is infinitely divisible? Use probability generating functions.

The sinc formula:

\[ \sum_{j=1}^{m} \varepsilon_j = \prod_{j=1}^{m} \varepsilon_j \quad \text{independent} \]

\[ \mathbb{P}(s) = \mathbb{E}_{s} \sum_{j=1}^{m} \varepsilon_j = \prod_{j=1}^{m} \mathbb{P}(s) \]

So one can check whether the family of probability distributions being referred to in a stochastic model satisfies this infinite divisibility condition. For a Poisson distribution, we'd want \( Y_j \sim \text{Poi}(\lambda_{B_j}) \) and in any case, we would need that \( \mu_j = \mathbb{E}Y_j \) would have to satisfy

\[ \mu = \mathbb{E}Y = \sum_{j=1}^{m} \mu_j. \]
\[ P(s) = \sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!} = e^{-(s-1)\lambda} \]

Infinite divisibility:

\[ \prod_{j=1}^{n} \mathbb{P}_j(s) = e^{-(s-1)\sum_{j=1}^{n} \lambda_j} = e^{-(s-1)\mu} \]